PUB. IRMA, LILLE 2009
Vol. 69, N° III

Lamperti's invariance principle for weak dependent sequences *†

Farida Achemine and Djamel Hamadouche
Laboratory of Mathematics
Faculty of Sciences
University M. Mammeri
Tizi-Ouzou, Algeria

acheminef@yahoo.fr, djhamad@yahoo.fr

Résumé

On considère le principe d'invariance de Lamperti pour des variables aléatoires vérifiant la condition de dépendance faible introduite par Doukhan et Louhichi. En utilisant quelques inégalités de moments établies, on montre une version du principe d'invariance de Lamperti pour les lignes polygonales d'interpolation du processus de sommes partielles. Le même résultat est établi pour le lissage par convolution du processus de sommes partielles.

Abstract

We consider Lamperti's invariance principle for random variables satisfying Doukhan-Louhichi dependence condition. With some moment inequalities, we obtain a version of Lamperti's invariance principle for the polygonal interpolation of the partial sums process. Similar results are proved for the convolution smoothing of partial sums process.

Keywords: Tightness, Hölder space, invariance principle, Brownian motion, weak dependence.
AMS classifications: 60B10, 60F05, 60F17, 62G30

---

*Preprint

†This paper was completed during the visit of the second author at University Lille 1 in March 2009

1 Introduction

We consider a sequence of real random variables $(X_n)_{n\geq 1}$. Set $\sigma^2 = \mathbb{E}X_1^2 + \sum_{k\geq 2}\mathbb{E}(X_1X_k)$ and $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. We assume there exist $\gamma > 2$ and $M > 0$ such that for all integers $n \geq 1$ and $h \geq 0$,

$$\mathbb{E}|X_n|^\gamma < M \quad \text{and} \quad \mathbb{E}|\sum_{k=1}^{n} X_{k+h}|^\gamma = \mathbb{E}|\sum_{k=1}^{n} X_k|^\gamma. \tag{1}$$

We denote ξ_n the random polygonal lines obtained by linear interpolation between the points $\left(\frac{j}{n}, \frac{S_j}{\sigma\sqrt{n}}\right)$.
When the X_j are identically distributed with $EX_j^2 = \sigma^2$, the Donsker-Prohorov's invariance principle establishes then the $C[0,1]$ weak convergence of ξ_n to the Brownian motion W. The invariance principle in the Banach Hölder space $H_\alpha[0,1]$ has been established by Lamperti [12]. Kerkyacharian and Roynette [11] have derived it again using the Faber-Schauder basis of triangular functions.

Theorem 1 ([12]): Let $(X_j)_{j\geq 1}$ be a sequence of independent identically distributed random variables with $EX_j = 0$ and $E|X_j|^2 = \sigma^2$. Suppose that for some constant $\gamma > 2$, $E|X_j|^\gamma < \infty$.

For all $n \in N^*$, $0 \leq j < n$, define

$$\xi_n(t,\omega) = \frac{1}{\sigma\sqrt{n}}\left[\sum_{k=1}^{j} X_k(\omega) + (nt - j)X_{j+1}(\omega)\right], \quad \frac{j}{n} \leq t < \frac{j+1}{n}.$$

Then the sequence $(\xi_n)_{n\geq 1}$ converges weakly to the Brownian motion W in H_α^0 for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$.

Using some results of tightness proved in these spaces, Hamadouche [10] has extended this result to dependent random variables (α-mixing and association) and has proved the weak convergence in H_α^0 of the convolution smoothed process to the Brownian motion. For statistical applications of the weak Hölder convergence (see e.g. [14]) where Suquet and Račkauskas obtained test statistics for epidemic change in Hölder norm.
Our purpose in this paper is to extend these results to the case of random variables under the weak dependence condition introduced by Doukhan and Louhichi [6]. It has been shown that this concept is more general than mixing and the association and contains lots of models of interest in statistics and econometrics (see e.g. [1]).
This concept is studied by several authors. The first invariance principle in $D[0,1]$ is given by Doukhan and Louhichi (1999) under existence of moments with order greater than 4. Thereafter, this condition has been improved by Dedecker and Doukhan (2003) under $\theta-$weak dependence, by Bulinski and Shashkin (2003) in the case of $k'-$weak dependence and by Doukhan and Wintenberger (2007) in

the case of k and λ-weak dependence, under the existence of moments with order greater than 2. Moments inequalities were established in the references [6, 9 and 13]. Several models and applications to the econometrics were studied in [1, 6 and 9]. Note that a complete reference on this topic is Dedecker and al [5].
The paper is organized as follows. Some weak dependence tools are presented in section 2. We recall the Hölderian framework and main results of weak convergence and tightness in section 3. In Section 4, we first give sufficient conditions for moment inequality. Using this result, we prove the Hölderian weak convergence of the polygonal and the convolution smoothed partial sums process under some weak dependence notions. These results rely essentially on moments inequalities and limit theorems.

2 Weak dependence tools

Doukhan and Louhichi (1999) introduced a new weak dependence which is more general than the strong mixing and the association.
Definition 1 ([6]): A sequence $(X_n)_{n\in\mathbb{Z}}$ is $(\epsilon, \mathbb{F}, \Psi)-$weakly dependent, if there exists a class of real-valued functions $\mathbb{F}$, a sequence $\epsilon = (\epsilon_r)_{r\in\mathbb{N}}$ decreasing to zero at infinity, and a function $\Psi : \mathbb{F}^2 \times \mathbb{N}^2 \to \mathbb{R}_+$ such that for any $(u+v)$-uplet $(i_1, i_2, ..., i_u, j_1, j_2, ..., j_v)$, with $i_1 \le i_2 \le ... \le i_u < i_u + r \le j_1 \le j_2 \le ... \le j_v$

$$|Cov(h(X_{i_1}, ..., X_{i_u}), k(X_{j_1}, ..., X_{j_v})| \le \Psi(h, k, u, v)\epsilon_r,$$

for all $h, k \in \mathbb{F}$ that are defined respectively on $\mathbb{R}^u$ and $\mathbb{R}^v$.

Remark 1. Specific function ψ gives different notions of weak dependence appropriate to describe and to study classes of important models in statistics and econometrics [3],[4] and [9]. We set in the definition 1,
$\mathbb{F} = \pounds_1 = \{f : \mathbb{R}^u \to \mathbb{R}, \text{measurable}; ||f||_\infty \le 1 \text{ and } Lip(f) < \infty\}$,
where $\|h\|_\infty = \sup\limits_{x\in\mathbb{R}^u} |h(x)|$ and $Lip(h) = \sup\limits_{x\neq y} \frac{|h(x)-h(y)|}{||x-y||_1}$ denotes the lipschitz modulus of continuity of h, with $||x||_1 = \sum\limits_{i=1}^{u} |x_i|$ for $x = (x_1, ..., x_u)$.
The sequence $(X_n)_{n\ge1}$ is
- $\lambda-$weakly dependent if $\Psi(h, k, u, v) = uvLip(h)Lip(k) + u\ Lip(h) + vLip(k)$ and we denote $\epsilon_r = \lambda_r$.
- $k-$weakly dependent if $\Psi(h, k, u, v) = uvLip(h)Lip(k)$ and we denote $\epsilon_r = k_r$.
- $k'-$weakly dependent if $\Psi(h, k, u, v) = vLip(h)Lip(k)$ and we denote $\epsilon_r = k'_r$.
- $\theta-$ weakly dependent if $\Psi(h, k, u, v) = vLip(k)$ and we denote $\epsilon_r = \theta_r$.

Remark 2. A sequence $k-$ weakly dependent is $\lambda-$weakly dependent [8].
More generally, we have
Remark 3. For two classes $\mathbb{F}_1$ and $\mathbb{F}_2$ of functions and two functions ψ_1 and ψ_2 (all defined as in the definition 1) such that $\mathbb{F}_1 \subset \mathbb{F}_2$ and $\psi_2 \le \psi_1$. A sequence of random variables $(X_n)_{n\ge1}$ $(\epsilon, \mathbb{F}_2, \psi_2)-$weakly dependent is $(\epsilon, \mathbb{F}_1, \psi_1)-$weakly dependent.

Indeed, let $(X_n)_n$ be $(\epsilon, \mathbb{F}_2, \psi_2)-$weakly dependant, then for all $(u+v)$-uplet $(i_1,...,i_u,j_1,...,j_v)$ with $i_1 \leq ... \leq i_u < i_u + r \leq j_1 \leq ... \leq j_v$, we have

$$|Cov(h(X_{i_1},...,X_{i_u}), k(X_{j_1},....,X_{j_v})| \leq \psi_2(h,k,u,v)\epsilon_r$$

for all $h, k \in \mathbb{F}_2$ that are definite respectively on $\mathbb{R}^u$ and $\mathbb{R}^v$.

Since $\mathbb{F}_1 \subset \mathbb{F}_2$, then $h, k \in \mathbb{F}_2$. Consequently

$$|Cov(h(X_{i_1},...,X_{i_u}), k(X_{j_1},...,X_{j_v})| \leq \psi_2(h,k,u,v)\ \epsilon_r$$

and with $\psi_2 \leq \psi_1$, we get $|Cov(h(X_{i_1},...,X_{i_u}), k(X_{j_1},...,X_{j_v})| \leq \psi_1(h,k,u,v)\ \epsilon_r$.

The sequence $(X_n)_{n\geq 1}$ is $(\epsilon, \mathbb{F}_1, \psi_1)-$weak dependent.

Examples
In the following, we present examples of sequences fulfilling this notion of weak dependence. For more examples we send back to the references [1],[7],[8] and [9]. We set $\pounds = \{h : \mathbb{R}^u \to \mathbb{R}; Lip(h) < \infty, \|h\|_\infty < \infty\}$.

• A associated stationary sequence $(X_n)_{n\geq 1}$ of centered random variables is λ-weakly dependent with $\lambda_r = O(\sup_{i\geq r}|Cov(X_0, X_i)|)$ [6].

• If a sequence $(X_n)_{n\geq 1}$ is strongly mixing, then it is $(\epsilon, \pounds_1, \Psi)-$weakly dependent with $\epsilon_n = \alpha_n$ and $\Psi(h,k,u,v) = 4||h||_\infty||k||_\infty$, where (α_n) are the mixing coefficients [6].

• A sequence $(X_n)_{n\in\mathbb{N}}$ of real-valued random variables defined on probability space $(\Omega, F, \mathbb{P})$ is said to be negatively associated if for any finite sets $I, J \subset \mathbb{N}$ such that $I \cap J = \emptyset$ and any coordinatewise non-decreasing functions $f : \mathbb{R}^{|I|} \longrightarrow \mathbb{R}$, $g : \mathbb{R}^{|J|} \longrightarrow \mathbb{R}$, $Cov(f(X_i, i \in I), g(X_j, j \in J)) \leq 0$,
whenever the covariance exists ($|I|$ denote the cardinality of finite set I).

A sequence $(X_n)_{n\in\mathbb{N}}$ of real-valued random variables and negatively associated with $\mathbb{E}X_n^2 < \infty$, is $(\theta, \pounds, \psi)$-weakly dependent with $\theta_n = \sup_{|i-j|\geq n}|cov(X_i, X_j|$ and $\Psi(h,k,u,v) = Lip(h)Lip(k)uv$.
Indeed, in Bulinski and Shabanovich (1998), it was shown that any negatively associated sequence of random variables with finite second moments satisfies for all finite subsets $I, J \subset \mathbb{N}$ such that $I \cap J = \emptyset$ and any bounded Lipschitz functions
$f : \mathbb{R}^{|I|} \longrightarrow \mathbb{R}$, $g : \mathbb{R}^{|J|} \longrightarrow \mathbb{R}$, the following inequality
$|Cov(f(X_i, i \in I), g(X_j, j \in J))| \leq Lip(f)Lip(g)\sum_{i\in I}\sum_{j\in J}|Cov(X_i; X_j)|$.
So for all $(u+v)$-uplet $(i_1, i_2, ..., i_u, j_1, j_2, ..., j_v)$, with $i_1 \leq i_2 \leq ... \leq i_u < i_u + r \leq j_1 \leq j_2 \leq ... \leq j_v$, we set $I = \{i_1, i_2, ..., i_u\}$, $J = \{j_1, j_2, ..., j_v\}$, then for any $h, k \in \pounds$ that are defined respectively on $\mathbb{R}^u$ and $\mathbb{R}^v$

$|Cov(h(X_{i_1},...,X_{i_u}),k(X_{j_1},...,X_{j_v})| \leq Lip(h)Lip(k)\sum_{i\in I}\sum_{j\in J}|Cov(X_i,X_j)|$
$\leq Lip(h)Lip(k)\sup_{|i-j|\geq r}|Cov(X_i,X_j)|uv.$

• Let (ε_n) be a sequence of independent and identically distributed random variables with $\mathbb{P}(\varepsilon_n=0)=\mathbb{P}(\varepsilon_n=1)=\frac{1}{2}$. The stationary Markov chain, $X_{n+1}=(X_n+\varepsilon_{n+1})/2$ with uniform marginal distribution on $[0,1]$, which is not mixing, is $(\epsilon,\pounds,\Psi)$-weakly dependent with $\epsilon_r=O(2^{-r})$ and $\Psi(h,k,u,v)=v||h||_\infty Lip(k)$.

• A stationary Gaussian process $(X_n)_{n\geq 1}$ such that $\underset{n\longrightarrow\infty}{lim} Cov(X_0,X_n)=0$ is $\lambda-$weakly dependent with $\lambda_r=O(\sup_{i\geq r}|Cov(X_0,X_i)|)$ [6].

• Let $(\varepsilon_n)_n$ be a sequence of i.i.d. and centered random variables. The stationary linear process and not causal given by $X_n=\sum_{j=-\infty}^{+\infty}b_j\varepsilon_{n-j}, n\in\mathbb{Z}$, with $b_j=O(|j|^{-\omega})$ and $\omega>\frac{1}{2}$ is $\lambda-$weakly dependent with $\lambda_r=O(r^{-\omega+\frac{1}{2}})$ [7].

• Let $(\varepsilon_n)_n$ be a sequence of i.i.d. and centered random variables. The linear and causal process given by $X_n=\sum_{j=0}^{+\infty}b_j\varepsilon_{n-j}, n\in\mathbb{Z}$, with $b_j=O(|j|^{-\mu})$ and $\mu>\frac{1}{2}$ is $\theta-$weakly dependent with $\theta_r=O(r^{-\mu+\frac{1}{2}})$ [7].

Definition 2 ([6]): Let $(X_n)_{n\geq 0}$ be a sequence of centered random variables. For positive integer r, the coefficient of weak dependence is the non-decreasing sequences $(C_{r,q})_{q\geq 2}$ such that

$$C_{r,q}=Sup|Cov(X_{t_1}...X_{t_m},X_{t_{m+1}}...X_{t_q})|,$$

where the supremum is taken over all $(t_1,....,t_q)$ such that $1\leq t_1\leq....\leq t_q$ and $t_{m+1}-t_m=r$.

Theorem 2 ([6]): Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of centered real random variables, fulfilling for some fixed $q\in\mathbb{N}$, $q\geq 2$, $C_{r,q}=O(r^{\frac{-q}{2}})$. Then there exists a positive constante C not depending on n such that

$$\mathbb{E}(|S_n|^q)\leq Cn^{\frac{q}{2}}.$$

We recall the following result due to Doukhan and Newmann (2007).
Lemma 1 ([9]): Let $(X_n)_{n\geq 1}$ be a sequence of real random variables $k-$weakly dependent (respectively $\lambda-$weakly dependent) and suppose there exists $m>q$ such that $E|X_i|^m\leq M_m$, for all $i\geq 1$. Then

$$C_{r,q}\leq q^4 2^{q+3}M_m^{\frac{q-1}{m-1}}k_r^{1-\frac{q-2}{m-2}},$$

(respectively $$C_{r,q}\leq q^4 2^{q+3}M_m^{\frac{q-1}{m-1}}\lambda_r^{1-\frac{q-1}{m-1}}).$$

For a sequence $(X_n)_{n\geq 1}$ of random variables, set $\sigma^2 = \mathbb{E}X_1^2 + \sum_{k\geq 2}\mathbb{E}(X_1X_k)$. W denotes the standard Brownian motion and $S_{[nt]} = \sum_{i=1}^{[nt]} X_i, \quad t \in [0,1], \quad n \in \mathbb{N}$. The Donsker invariance principle $\frac{1}{\sqrt{n}} S_{[nt]} \underset{n\longrightarrow\infty}{\longrightarrow} \sigma W(t)$, in distribution in the Skorohod space $D[0,1]$, under λ and k weak dependence conditions is obtained by Doukhan and Wintenberger [8]. Under k'-weak dependence, the result is proved by Bulinski and Shashkin [2]. Dedeker and Doukhan obtained this result under θ-weak dependence [4]. We recall that in the following result.

Theorem 3: Let $(X_n)_{n\geq 1}$ be a stationary sequence of centered random variables. Suppose there exists $m > 2$ such that $\mathbb{E}|X_1|^m < \infty$. Then $\sigma^2 = \mathbb{E}X_1^2 + \sum_{k\geq 2}\mathbb{E}(X_1X_k)$ is well defined and $(\frac{1}{\sqrt{n}} S_{[nt]})$ converges weakly to σW in $D[0,1]$ if one of the following additional assumptions is fulfilled:
• the sequence is $\theta-$weakly dependent with $\theta_r = O(r^{-\theta})$ (as $r \uparrow \infty$) for $\theta > 1 + \frac{1}{m-2}$.
• the sequence is $k'-$weakly dependent with $k'_r = O(r^{-k})$ (as $r \uparrow \infty$) for $k > 1 + \frac{1}{m-2}$.
• the sequence is $\lambda-$weakly dependent with $\lambda_r = O(r^{-\lambda})$ (as $r \uparrow \infty$) for $\lambda > 4 + \frac{2}{m-2}$.
• the sequence is $k-$weakly dependent with $k_r = 0(r^{-k})$ (as $r \uparrow \infty$) for $k > 2 + \frac{1}{m-2}$.

3 The Hölderian framework $H_\alpha[0,1]$

3.1 Definitions

We use the notations and results of Ciesielski about the space of Hölderian functions on $[0,1]$.
We define the Hölder space $H_\alpha[0,1]$ $(0 \leq \alpha \leq 1)$ as the space of functions f vanishing at 0 such that:

$$\|f\|_\alpha = \sup_{|t-s|\leq 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < +\infty.$$

We denote $\omega_\alpha(f,\delta)$ the Hölderian modulus of continuity of f

$$\omega_\alpha(f,\delta) = \sup_{|t-s|\leq \delta} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

The subspace $H_\alpha^0[0,1]$ of $H_\alpha[0,1]$ is defined by

$$f \in H_\alpha^0[0,1] \Longleftrightarrow f \in H_\alpha[0,1] \text{ and } \lim_{\delta\to 0} \omega_\alpha(f,\delta) = 0.$$

$(H_\alpha, \|.\|_\alpha)$ is a non-separable Banach space. For $0 < \beta < \alpha$, $(H_\alpha, \|.\|_\beta)$ is separable and H_α is topologically embedded in H_β. $(H_\alpha^0, \|.\|_\alpha)$ is a separable closed subspace of $(H_\alpha, \|.\|_\alpha)$.

3.2 Random element in H_α

In this section, we consider stochastic process with Hölderian paths as random element ξ of the functional space H_α and we denote P_ξ its distribution. The study of weak convergence of random element of H_α^0 is based on the following result.

Proposition 1 ([10]): *The weak convergence in H_α^0 of a sequence of processes $\{\xi_n, n \geq 1\}$ is equivalent to the tightness in H_α^0 of the sequence of distributions $P_n = P\xi_n^{-1}$ of random elements ξ_n and the convergence of the finite-dimensional distributions of ξ_n.*

3.3 Tightness in H_α^0

To prove the weak convergence, we need tightness. For more convenience, we work in $H_\alpha^0[0,1]$ which is separable instead of $H_\alpha[0,1]$. As the canonical injection of H_α^0 in H_α is continous, weak convergence in the former implies weak convergence in the later. A first sufficient condition of tightness is given by

Theorem 4 ([11]): *Let $(\xi_n)_{n\geq 1}$ be a sequence of processes vanishing at 0 and suppose that there are $\delta > 0, \gamma > 0$ and $c > 0$ such that:*

$$\forall \lambda > 0, \quad P(|\xi_n(t) - \xi_n(s)| > \lambda) \leq \frac{c}{\lambda^\gamma} |t-s|^{1+\delta}.$$

Then the sequence $(\xi_n)_{n\geq 1}$ is tight in $H_\alpha^0[0,1]$ for $0 < \alpha < \frac{\delta}{\gamma}$.

By the Markov's inequality, we have the moments version of the last theorem.

Corollary 1 ([12]): *Let $(\xi_n)_{n\geq 1}$ be a sequence of processes vanishing at 0. Suppose there are $\delta > 0, \gamma > 0$ and $c > 0$ such that*

$$E|\xi_n(t) - \xi_n(s)|^\gamma \leq c|t-s|^{1+\delta}.$$

Then the sequence $(\xi_n)_{n\geq 1}$ is tight in $H_\alpha^0[0,1]$ for $0 < \alpha < \frac{\delta}{\gamma}$.

For more flexibility in the handling of moment inequalities, we use the following theorem.

Theorem 5 ([10]):*Let $(\xi_n)_{n\geq 1}$ be a sequence of random elements of $H_\alpha^0[0,1]$, satisfying the following conditions:*

a) There exists constants $a > 1, b > 1, c > 0$ and a sequence of positive numbers $(a_n) \searrow 0$ such that

$$E|\xi_n(t) - \xi_n(s)|^a \leq c|t-s|^b,$$

for all $|t-s| \geq a_n, 0 \leq s,t \leq 1$ and $n \geq 1$.

b) For any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\{\omega_\alpha(\xi_n, a_n) > \varepsilon\} = 0.$$

Then for all $\alpha < a^{-1}(\min(a,b) - 1)$, $(\xi_n)_{n\geq 1}$ is tight in $H_\alpha^0[0,1]$.

4 Invariance principles in H_α

We consider a sequence of real random variables λ, k, k' or θ-weak dependent and we prove the Hölderian weak convergence to the Brownian motion of the polygonal and the convolution smoothed partial sums process. The proof rely essentially on limit theorems and Lemma given bellow.

4.1 Moment inequality

Lemma 2: Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of centered real random variables such that $\mathbb{E}|X_t|^m \le M_m$, for some $m > q \ge 2$ and $M_m > 0$. Assume that the sequence $(X_n)_{n\ge 1}$ is either $\lambda-$weak dependent satisfying $\lambda_r = O(r^{-\nu}),\ \nu \ge \frac{q}{2}(\frac{m-1}{m-q})$ or $k-$weak dependent satisfying $k_r = O(r^{-\nu}),\ \nu \ge \frac{q}{2}(\frac{m-2}{m-q})$. Then there exists a positive constante C not depending on n such that

$$\mathbb{E}(|S_n|^q) \le Cn^{\frac{q}{2}}. \tag{2}$$

Proof. Without less of generallity, we do the proof under λ-weak dependence and the proof in the case of k-weak dependence is similar.
Under the conditions of this lemma, we have (see Lemma 1)

$$C_{r,q} \le q^4 2^{q+3} M_m^{\frac{q-1}{m-1}} \lambda_r^{1-\frac{q-1}{m-1}},\ \forall r \ge 0.$$

Then

$$C_{r,q} = O(\lambda_r^{1-\frac{q-1}{m-1}}).$$

Using the condition $\lambda_r = O(r^{-\nu}),\ \nu \ge \frac{q}{2}\frac{m-1}{m-q}$, we get $C_{r,q} = O(r^{\frac{-q}{2}})$ and we conclude by theorem 2.

Remark 4.
1) Note that a sequence of real random variables θ-weak dependent is λ-weak dependent. Then, under θ-weak dependence, this Lemma remains true with $\theta_r = O(r^{-\nu}),\ \nu \ge \frac{q}{2}(\frac{m-1}{m-q})$.
2) Also, a sequence of real random variables k'-weak dependent is k-weak dependent. Then, under k'-weak dependence, this Lemma remains true with $k'_r = O(r^{-\nu}),\ \nu \ge \frac{q}{2}(\frac{m-2}{m-q})$.

4.2 Polygonal smoothed partial sums process

For a sequence $(X_n)_{n\ge 1}$of real random variables, set $\sigma^2 = \mathbb{E}X_1^2 + \sum_{k\ge 2}\mathbb{E}(X_1X_k)$ and $S_n = \sum_{i=1}^{n} X_i, n \ge 1$. We assume that the condition (1) is fulfilled.

For all $n \in N^*$, $0 \leq j < n$, define

$$\xi_n(t,\omega) = \frac{1}{\sigma\sqrt{n}}\left[\sum_{k=1}^{j} X_k(\omega) + (nt-j)X_{j+1}(\omega)\right], \quad \frac{j}{n} \leq t < \frac{j+1}{n}.$$

Theorem 6: Let $(X_n)_{n\geq 1}$ be a stationary sequence of centered real random variables fulfilling the condition (1) and for some $m > \gamma + 2$, $E|X_n|^m < \infty$. Assume that the sequence $(X_n)_{n\geq 1}$ is either $\lambda-$weak dependent, satisfying

$$\lambda_r = O(r^{-\lambda}), \quad for \quad \lambda > max(4 + \frac{2}{m-2}, \frac{([\frac{\gamma}{2}]+1)(m-1)}{m-\gamma-2}),$$

either $k-$weak dependent, with

$$k_r = O(r^{-k}), \quad for \quad k > max(2 + \frac{1}{m-2}, \frac{([\frac{\gamma}{2}]+1)(m-2)}{m-\gamma-2}),$$

either $k'-$weak dependent, with

$$k'_r = O(r^{-k}), \quad for \quad k > max(1 + \frac{1}{m-2}, \frac{([\frac{\gamma}{2}]+1)(m-2)}{m-\gamma-2}),$$

or $\theta-$weak dependent, with

$$\theta_r = O(r^{-\theta}), \quad for \quad \theta > max(1 + \frac{1}{m-2}, \frac{([\frac{\gamma}{2}]+1)(m-1)}{m-\gamma-2}).$$

Then the sequence $(\xi_n)_{n\geq 1}$ converges weakly to the Brownian motion W in H^0_α, for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$.

Proof.
Convergence of finite-dimensional distributions of $(\xi_n)_{n\geq 1}$.
The conditions of theorem 3 (according to the notion of weak dependence) are fulfilled. Consequently, the finite-dimensionnal distributions of $W_n(t) = \frac{1}{\sigma\sqrt{n}}S_{[nt]}$ converge to those of the Brownian motion W. Then it suffices to prove that the distance in $\mathbb{R}^k$, between $(\xi_n(t_1),...,\xi_n(t_k))$ and $(W_n(t_1),...,W_n(t_k))$ goes to zero in probability, for all $k \in \mathbb{N}^*$ and $0 \leq t_1 \leq t_2 \leq ... \leq t_k \leq 1$.
We consider the Euclidian norm on $\mathbb{R}^k$: $||\xi_n - W_n||^2_{\mathbb{R}^k} = \sum_{i=1}^{k}|\xi_n(t_i) - \frac{1}{\sigma\sqrt{n}}S_{[nt_i]}|^2$.
Since $\xi_n(t) - \frac{1}{\sigma\sqrt{n}}S_{[nt]} \xrightarrow{L^2} 0$ implies $\xi_n(t) - \frac{1}{\sigma\sqrt{n}}S_{[nt]} \underset{n\to\infty}{\xrightarrow{\mathbb{P}}} 0$, then it is sufficient to prove that $\xi_n(t) - \frac{1}{\sigma\sqrt{n}}S_{[nt]} \xrightarrow{L^2} 0, \quad \forall t \in [0,1]$.
Let $t \in [0,1]$, there exists $j \in \mathbb{N}$ such that $\frac{j}{n} \leq t < \frac{j+1}{n}$ and

$$\mathbb{E}|\xi_n(t) - \frac{1}{\sigma\sqrt{n}}S_{[nt]}|^2 = \mathbb{E}|\frac{1}{\sigma\sqrt{n}}[\sum_{i=1}^{j}X_i + (nt-j)X_{j+1}] - \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{[nt]}X_i|^2$$

$$\begin{aligned} &= \mathbb{E}|\tfrac{1}{\sigma\sqrt{n}}(nt-[nt])X_{[nt]+1}|^2 \\ &\leq |\tfrac{1}{\sigma^2 n}(nt-[nt])^2|\mathbb{E}X_{[nt]+1}|^2 \\ &\leq \tfrac{1}{\sigma^2 n}\mathbb{E}|X_{[nt]+1}|^2, \text{ since } (nt-[nt])^2 \leq 1. \end{aligned}$$

Using the Hölder inequality for $\mathbb{E}|X_{[nt]+1} \times 1|^2$, we get

$$\mathbb{E}|\xi_n(t) - \frac{1}{\sigma}S_{[nt]}|^2 \leq \frac{1}{\sigma^2 n}\ (\mathbb{E}(|X_{[nt]+1}|^\gamma))^{\frac{2}{\gamma}} \leq \frac{1}{\sigma^2 n}M^{\frac{2}{\gamma}}.$$

Then, we obtain

$$\xi_n(t) - \tfrac{1}{\sigma\sqrt{n}}S_{[nt]} \xrightarrow{L^2} 0, \forall t \in [0,1].$$

It follows that

$$\xi_n(t) - \tfrac{1}{\sigma\sqrt{n}}S_{[nt]} \underset{n\to\infty}{\xrightarrow{\mathbb{P}}} 0, \forall t \in [0,1].$$

Consequently

$$||\xi_n(t) - \tfrac{1}{\sigma\sqrt{n}}S_{[nt]}||^2_{\mathbb{R}^k} = \sum_{i=1}^{k}|\xi_n(t_i) - \tfrac{1}{\sigma\sqrt{n}}S_{[nt_i]}|^2 \xrightarrow{\mathbb{P}} 0, \text{ as } n \text{ goes to } \infty.$$

Tightness.
We prove that under the assumptions of theorem 6:
$\mathbb{E}|\xi_n(t) - \xi_n(s)|^\gamma \leq B|t-s|^{\frac{\gamma}{2}}$, where B is a positive constant.
Let $s, t \in [0,1]$, we consider two cases:

- $\frac{j}{n} \leq s \leq t \leq \frac{j+1}{n}$.

$$\begin{aligned} \mathbb{E}|\xi_n(t) - \xi_n(s)|^\gamma &= |\tfrac{1}{\sigma\sqrt{n}}[\sum_{i=1}^{j}X_i + (nt-j)X_{j+1}] - \tfrac{1}{\sigma\sqrt{n}}[\sum_{i=1}^{j}X_i + (ns-j)X_{j+1}]|^\gamma \\ &= (\tfrac{1}{\sigma\sqrt{n}})^\gamma(nt-ns)^\gamma\ \mathbb{E}|X_{j+1}|^\gamma \\ &\leq (\tfrac{1}{\sigma})^\gamma(n(t-s))^{\frac{\gamma}{2}}(t-s)^{\frac{\gamma}{2}}\ \mathbb{E}|X_{j+1}|^\gamma \\ &\leq (\tfrac{1}{\sigma})^\gamma(t-s)^{\frac{\gamma}{2}}\ \mathbb{E}|X_{j+1}|^\gamma, \text{ since } n(t-s) \leq 1. \end{aligned}$$

Then, for $B_1 = (\frac{1}{\sigma})^\gamma M$, we obtain

$$\mathbb{E}|\xi_n(t) - \xi_n(s)|^\gamma \leq B_1|t-s|^{\frac{\gamma}{2}}. \tag{3}$$

- $\frac{j-1}{n} \leq s \leq \frac{j}{n} \leq ... \leq \frac{j+k}{n} \leq t \leq \frac{j+k+1}{n}$.

By Jensen inequality, we have

$$\begin{aligned} &\mathbb{E}|\xi_n(t) - \xi_n(s)|^\gamma = \mathbb{E}|\xi_n(t) - \xi_n(\tfrac{j+k}{n}) + \xi_n(\tfrac{j+k}{n}) - \xi_n(\tfrac{j}{n}) + \xi_n(\tfrac{j}{n}) - \xi_n(s)|^\gamma \\ &\quad \leq 3^{\gamma-1}(\mathbb{E}|\xi_n(t) - \xi_n(\tfrac{j+k}{n})|^\gamma + \mathbb{E}|\xi_n(\tfrac{j+k}{n}) - \xi_n(\tfrac{j}{n})|^\gamma + \mathbb{E}|\xi_n(\tfrac{j}{n}) - \xi_n(s)|^\gamma). \end{aligned}$$

As previously, we show that

$\exists B_1, B_2 \in \mathbb{R}_+^*$ such that

$$\mathbb{E}|\xi_n(t) - \xi_n(\frac{j+k}{n})|^\gamma \leq B_2|t-s|^p \text{ and } \mathbb{E}|\xi_n(\frac{j}{n}) - \xi_n(s)|^\gamma \leq B_3|t-s|^{\frac{\gamma}{2}}. \quad (4)$$

For the middle term, by using (1), we get

$\mathbb{E}|\xi_n(\frac{j+k}{n}) - \xi_n(\frac{j}{n})|^\gamma = \mathbb{E}|\frac{1}{\sigma\sqrt{n}}\sum_{i=j+1}^{j+k} X_i|^\gamma = \mathbb{E}|\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{k} X_i|^\gamma.$

By Hölder's inequality

$\mathbb{E}|\xi_n(\frac{j+k}{n}) - \xi_n(\frac{j}{n})|^\gamma \leq (\mathbb{E}|\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{k} X_i|^{\gamma+2})^{\frac{\gamma}{\gamma+2}}.$

The conditions of lemma 2 are satisfied under λ and k-weak dependence and see Remark 4, for θ and k'-weak dependence. Then by using (2) with $q = 2[\frac{\gamma}{2}] + 2$, we get

$$\begin{aligned}
\mathbb{E}|\frac{1}{\sigma\sqrt{n}}\sum_{i=j+1}^{j+k} X_i|^{\gamma+2} &\leq (\frac{1}{\sigma\sqrt{n}})^{\gamma+2} C k^{[\frac{\gamma}{2}]+1} \\
&\leq (\frac{1}{\sigma})^{\gamma+2} C (\frac{k}{n})^{[\frac{\gamma}{2}]+1} \\
&\leq (\frac{1}{\sigma})^{\gamma+2} C |t-s|^{[\frac{\gamma}{2}]+1}, \text{ since } \frac{k}{n} \leq t-s.
\end{aligned}$$

Hence

$\mathbb{E}|\xi_n(\frac{j+k}{n}) - \xi_n(\frac{j}{n})|^\gamma \leq ((\frac{1}{\sigma})^{\gamma+2} C |t-s|^{[\frac{\gamma}{2}]+1})^{\frac{\gamma}{\gamma+2}} = (\frac{1}{\sigma})^\gamma C^{\frac{\gamma}{\gamma+2}} |t-s|^{\frac{\gamma}{2}}.$

Finally

$$\mathbb{E}|\xi_n(\frac{j+k}{n}) - \xi_n(\frac{j}{n})|^\gamma \leq B_4|t-s|^{\frac{\gamma}{2}}, \quad \text{with} \quad B_4 = (\frac{1}{\sigma})^\gamma \; C^{\frac{\gamma}{2[\frac{\gamma}{2}]+2}}. \quad (5)$$

From (3) to (5), we deduce

$$\forall t, s \in [0,1], \; \mathbb{E}|\xi_n(t) - \xi_n(s)|^\gamma \leq B|t-s|^{1+\delta},$$

with $B = \max(B_1, 3^{\gamma-1}(B_2 + B_3 + B_4))$ and $\delta = \frac{\gamma}{2} - 1 > 0$.

By corollary 1, we conclude that $(\xi_n(t))_{n\geq 1}$ is tight in H_α^0, for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$. This achieves the proof of theorem 6.

4.3 Example

We give now an example which satisfies the condition (1) and the conditions of Lemma 2.

Let $\gamma > 2$ and $(\epsilon_n)_{n\in\mathbb{Z}}$ be a stationary sequence of real i.i.d. centered random variables with $\mathbb{E}|\epsilon_n|^\gamma < \infty$, for all integer n. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $\sum_{i=-\infty}^{+\infty} a_i\epsilon_i$ converges in L^γ. Then the sequence defined by

$X_n = \sum_{i=0}^{+\infty} a_i \epsilon_{n-i}, n \geq 1$, with $a_k = O(|k|^{-\mu})$, $\mu > \frac{1}{2}$

1) is $\lambda-$weak dependent with $\lambda_r = O(\frac{1}{r^{\mu-\frac{1}{2}}})$ (Doukhan and Lang, 2002),

2) satisfies conditions (2) (Miamee and Pourahmadi, 1988),

3) under the additional assumptions: $\mathbb{E}X_1 = 0$ and there exists $d > \gamma + 2$ such that $\mathbb{E}|X_n|^d < \infty$ and $\mu = \upsilon + \frac{1}{2}$, for $\upsilon > \frac{([\frac{\gamma}{2}]+1)(d-1)}{d-\gamma-2}$, we get

$$\lambda_r = O(r^{-\mu}).$$

By lemma 1, we obtain

$$C_{r,q} \leq q^4 2^{q+3} M_m^{\frac{q-1}{m-1}} \lambda_r^{1-\frac{q-1}{m-1}}.$$

Therefore

$$C_{r,\gamma+2} = O(r^{-[\frac{\gamma}{2}]-1}).$$

4.4 The convolution smoothed partial sums process

For a sequence $(X_n)_{n\geq 1}$ of real random variables, set $\sigma^2 = \mathbb{E}X_1^2 + \sum_{k\geq 2} \mathbb{E}(X_1 X_k)$ and $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. If σ^2 is finite and $\neq 0$, we consider the normalized partial sums process

$$W_n(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{[nt]} X_i, \quad t \in [0,1].$$

For the sake of convenience, we shall use the following expression of W_n,

$$W_n(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} X_i . 1_{[\frac{i}{n},1[}, \quad t \in [0,1].$$

We consider in the following, the convolution smoothed partial sums process which gives more regular paths than the polygonal one.
We first recall here some assumptions used by Hamadouche [10]. We define the sequence $(k_n)_{n\geq 1}$ of convolution kernels by

$$k_n(t) = \frac{1}{b_n} k(\frac{t}{b_n}), \quad t \in \mathbb{R}. \tag{6}$$

with k a probability density on the real line such that

$$\int_{\mathbb{R}} |u|k(u)du < +\infty, \tag{7}$$

and $(b_n)_{n\geq 1}$ a sequence of positive numbers such that $lim_{n\to\infty} b_n = 0$ and

$$\frac{1}{b_n} = O(n^{\frac{\tau}{2}}), \quad 0 < \tau < \frac{1}{2}. \tag{8}$$

The convolution smoothed partial sums process is defined by

$$\zeta_n(t) = (W_n * k_n)(t) - (W_n * k_n)(0), \quad t \in [0,1], \quad n \geq 1. \tag{9}$$

The term $(\xi_n * k_n)(0)$ is added in order to have a process with paths vanishing at 0.
Under the assumptions

$$k \ \in L^1([-1,1]) \cap L^{\frac{1}{2}}[-1,1]), \tag{10}$$

and

$$|k(x) - k(y)| \leq a(k)|x - y|, \tag{11}$$

for some constante $a(k)$, we have the following result.

Theorem 7: Let $(X_n)_{n\geq 1}$ be a sequence of centered real random variables which are λ, θ ,k' or $k-$weakly dependent. Assume that the conditions of theorem 6 are fulfilled and the convolution kernels satisfy (6), (7), (8),(10) and (11). Then the sequence $(\zeta_n)_{n\geq 1}$ defined by (9) converges weakly to the Brownian motion W in H^0_α for all $\alpha < \frac{1}{2} - \max(\tau, \frac{1}{\gamma})$.

Proof.
Convergence of finite-dimensional distributions of $(\zeta_n)_{n\geq 1}$.
As in the precedent case, by theorem 3, it suffices to prove the convergence to zero of $\mathbb{E}|W_n * k_n(t) - W_n(t)|^2$. Indeed, we have

$$\begin{aligned}
\mathbb{E}|W_n * k_n(t) - W_n(t)|^2 &= \mathbb{E}|\textstyle\int_{\mathbb{R}}[W_n(t-u)k_n(u) - W_n(t)]du|^2 \\
&= \mathbb{E}|\textstyle\int_{\mathbb{R}} W_n(t-u)k_n(u)du - \int_{\mathbb{R}} W_n(t)k_n(u)du|^2 \\
&= \mathbb{E}|\textstyle\int_{\mathbb{R}}(W_n(t-u) - W_n(t))k_n(u)du|^2 \\
&= \mathbb{E}|\textstyle\int_{\mathbb{R}} \frac{1}{\sigma\sqrt{n}}(S_{[n(t-u)]} - S_{[nt]})k_n(u)du|^2 \\
&= \mathbb{E}|\mathbb{E}_{k_n}[\frac{1}{\sigma\sqrt{n}}(S_{[n(t-u)]} - S_{[nt]})]|^2.
\end{aligned}$$

Where $\mathbb{E}_{k_n}$ is the expectation with respect to the probability measure $k_n(u)du$.

Using the Jensen's inequality

$$\mathbb{E}|W_n * k_n(t) - W_n(t)|^2 \le \mathbb{E}[\mathbb{E}_{k_n}|\frac{1}{\sigma\sqrt{n}}(S_{[n(t-u)]} - S_{[nt]})|^2].$$

Applying Fubini's theorem, we get

$$\mathbb{E}|W_n * k_n(t) - W_n(t)|^2 \le \mathbb{E}_{k_n}[\mathbb{E}|\tfrac{1}{\sigma\sqrt{n}}(S_{[n(t-u)]} - S_{[nt]})|^2]$$
$$\le \int_{\mathbb{R}}(\tfrac{1}{\sigma\sqrt{n}})^2\mathbb{E}|(S_{[n(t-u)]} - S_{[nt]})|^2 k_n(u)du.$$

On the other hand,

$$\mathbb{E}|(S_{[n(t-u)]} - S_{[nt]})|^2 \le (\mathbb{E}|(S_{[n(t-u)]} - S_{[nt]})|^\gamma)^{\frac{2}{\gamma}}.$$

Using condition (1), we have

$$\mathbb{E}|(S_{[n(t-u)]} - S_{[nt]})|^\gamma = \mathbb{E}|(S_{[n(t-u)]-[nt]})|^\gamma.$$

By Hölder's inequality,

$$\mathbb{E}|(S_{[n(t-u)]} - S_{[nt]})|^2 \le (\mathbb{E}|(S_{[n(t-u)]-[nt]})|^\gamma)^{\frac{2}{\gamma}} \le (\mathbb{E}|(S_{[n(t-u)]-[nt]})|^{\gamma+2})^{\frac{1}{[\frac{\gamma}{2}]+1}}.$$

The conditions of Lemma 2 are fulfilled for λ and k-weak dependence with $q = \gamma + 2$. They are also fulfilled for θ and k'-weak dependence (see Remark 4). Then we can use inequality (2) for these four notions of dependence. We get

$$\mathbb{E}|S_{([n(t-u)]-[nt])}|^2 \le (C([n(t-u)] - [nt])^{[\frac{\gamma}{2}]+1})^{\frac{1}{[\frac{\gamma}{2}]+1}},$$

where C is a positive constant.

Consequently

$$\mathbb{E}|W_n * k_n(t) - W_n(t)|^2 \le \int_{\mathbb{R}}(\tfrac{1}{\sigma\sqrt{n}})^2\ C^{\frac{1}{[\frac{\gamma}{2}]+1}}([n(t-u)] - [nt])k_n(u)du.$$

Thus, with $[n(t-u)] - [nt] \le n(|u| + \frac{2}{n})$, we get

$$\mathbb{E}|W_n * k_n(t) - W_n(t)|^2 \le \int_{\mathbb{R}} \tfrac{n}{\sigma^2 n} C^{\frac{1}{[\frac{\gamma}{2}]+1}}[|u| + \tfrac{2}{n}]k_n(u)du$$
$$\le C' \int_{\mathbb{R}}[|u| + \tfrac{2}{n}]k_n(u)du, \text{ with } C' = (\tfrac{1}{\sigma^2}C^{\frac{1}{[\frac{\gamma}{2}]+1}}).$$

By (6) and for $v = \frac{u}{b_n}$, it follows that

$$\mathbb{E}|W_n * k_n(t) - W_n(t)|^2 \le C' \int_{\mathbb{R}}[|vb_n| + \tfrac{2}{n}]k_n(vb_n)b_n dv$$

$$\le C' \int_{\mathbb{R}}[b_n|v| + \tfrac{2}{n}]\tfrac{1}{b_n}k(v)b_n dv$$

$$\le C'(b_n \int_{\mathbb{R}}|v|k(v)dv + \int_{\mathbb{R}}\tfrac{2}{n}k(v)dv$$

$$\leq C'(b_n \int_{\mathbb{R}} |v|k(v)dv + \tfrac{2}{n}).$$

Since $\int_{\mathbb{R}} |v|k(v)dv < \infty$ and $b_n \to 0$, as n goes to infinity, we deduce that $W_n * k_n(t) - W_n(t)$ converge to zero in L^2 for all $t \in [0,1]$. In particular for $t = 0$, $\mathbb{E}|W_n * k_n(0) - W_n(0)|^2 = \mathbb{E}|W_n * k_n(0)|^2$ goes to zero, as n goes to infinity. For $t \in [0,1]$, we have

$$\mathbb{E}|\zeta_n(t) - W_n(t)|^2 = \mathbb{E}|W_n * k_n(t) - W_n * k_n(0) - W_n(t)|^2.$$

So by Jensen's inequality, we get
$\mathbb{E}|\zeta_n(t) - W_n(t)|^2 \leq 2(\mathbb{E}|W_n * k_n(t) - W_n(t)|^2 + \mathbb{E}|W_n * k_n(0)|^2) \underset{n\longrightarrow\infty}{\longrightarrow} 0.$

From this inequality, we deduce that for all $t \in [0,1]$, $\zeta_n(t) - W_n(t) \overset{L^2}{\to} 0$, then $\zeta_n(t) - W_n(t) \overset{P}{\to} 0$. Consequently, for all $k \in \mathbb{N}^*$ and $0 \leq t_1 \leq t_2 \leq ... \leq t_k \leq 1$, we have

$$\sum_{i=1}^{k} |\zeta_n(t_i) - W_n(t_i)|^2 = ||\zeta_n - W_n||^2_{\mathbb{R}_k} \overset{\mathbb{P}}{\to} 0.$$

So the finite-dimensional distribution of ζ_n converges to those of W.

Tightness.
We apply theorem 5 with $a_n = \frac{1}{n}$.
Let $s, t \in [0,1]$ and assume that $t > s$ (without loss of generality). We consider two cases.

• $t - s \geq \frac{1}{n}$.
$\mathbb{E}|\zeta_n(t) - \zeta_n(s)|^\gamma = \mathbb{E}|(W_n * k_n)(t) - (W_n * k_n)(s)|^\gamma$

$$= \mathbb{E}|\int_{\mathbb{R}} (\tfrac{1}{\sigma\sqrt{n}} S_{[n(t-u)]} - \tfrac{1}{\sigma\sqrt{n}} S_{[n(s-u)]}) k_n(u)du|^\gamma$$

$$= \mathbb{E}|\int_{\mathbb{R}} (\tfrac{1}{\sigma\sqrt{n}} \sum_{i=[n(s-u)]+1}^{[n(t-u)]} X_i) k_n(u)du|^\gamma$$

$$= \mathbb{E}|\mathbb{E}_{k_n}(\tfrac{1}{\sigma\sqrt{n}} (\sum_{i=[n(s-u)]+1}^{[n(t-u)]} X_i)|^\gamma,$$

By Jensen's inequality, we obtain

$$\mathbb{E}|\zeta_n(t) - \zeta_n(s)|^\gamma \leq \mathbb{E}(\mathbb{E}_{k_n}|\tfrac{1}{\sigma\sqrt{n}} (\sum_{i=[n(s-u)]+1}^{[n(t-u)]} X_i)|^\gamma).$$

Using Fubini's theorem, we get

$$\mathbb{E}|\zeta_n(t) - \zeta_n(s)|^\gamma \leq \mathbb{E}_{k_n}(\mathbb{E}|\tfrac{1}{\sigma\sqrt{n}} (\sum_{i=[n(s-u)]+1}^{[n(t-u)]} X_i)|^\gamma)$$

$$\leq \int_{\mathbb{R}} (\frac{1}{\sigma\sqrt{n}})^\gamma \mathbb{E}|\sum_{i=[n(s-u)]+1}^{[n(t-u)]} X_i|^\gamma k_n(u)du.$$

By condition (1), we have

$$\mathbb{E}|\sum_{i=[n(s-u)]+1}^{[n(t-u)]} X_i|^\gamma = \mathbb{E}|\sum_{i=1}^{[n(t-u)]-[n(s-u)]]} X_i|^\gamma.$$

Then, using Hölder's inequality, we obtain

$$\mathbb{E}|\zeta_n(t)-\zeta_n(s)|^\gamma \leq \int_{\mathbb{R}} (\frac{1}{\sigma\sqrt{n}})^\gamma (\mathbb{E}|\sum_{i=1}^{[n(t-u)]-[n(s-u)]} X_i|^{\gamma+2})^{\frac{\gamma}{\gamma+2}} k_n(u)du.$$

The conditions of Lemma 2 are fulfilled for λ and k-weak dependence with $q=\gamma+2$. Using inequality (2), we get

$$\mathbb{E}|\sum_{i=1}^{[n(t-u)]-[n(s-u)]} X_i|^{\gamma+2} \leq C_1([n(t-u)]-[n(s-u)])^{[\frac{\gamma}{2}]+1}.$$

Consequently

$$\mathbb{E}|\zeta_n(t)-\zeta_n(s)|^\gamma \leq \int_{\mathbb{R}} (\frac{1}{\sigma\sqrt{n}})^\gamma \ C_1^{\frac{\gamma}{\gamma+2}} ([n(t-u)]-[n(s-u)])^{\frac{\gamma}{2}} k_n(u)du.$$

Since $[n(t-u)]-[n(s-u)] \leq n(t-s)+2$ and $\frac{1}{n} \leq |t-s|$, we obtain

$$\begin{aligned}
\mathbb{E}|\zeta_n(t)-\zeta_n(s)|^\gamma &\leq \frac{1}{(\sigma\sqrt{n})^\gamma} C_1^{\frac{\gamma}{\gamma+2}} (n(t-s)+2)^{\frac{\gamma}{2}} \\
&\leq \frac{1}{\sigma^\gamma n^{\frac{\gamma}{2}}} \ C_1^{\frac{\gamma}{\gamma+2}} n^{\frac{\gamma}{2}} ((t-s)+\frac{2}{n})^{\frac{\gamma}{2}} \\
&\leq \frac{1}{\sigma^\gamma} \ C_1^{\frac{\gamma}{\gamma+2}} (|t-s|+|t-s|2)^{\frac{\gamma}{2}} \\
&\leq (\frac{1}{\sigma^\gamma} \ C_1^{\frac{\gamma}{\gamma+2}} \ 3^{\frac{\gamma}{2}})|t-s|^{\frac{\gamma}{2}}.
\end{aligned}$$

So there is a constant $C_2 = (\frac{1}{\sigma^\gamma} C_1^{\frac{\gamma}{\gamma+2}} 3^{\frac{\gamma}{2}})$ such that for $t-s \geq \frac{1}{n}$, we have

$$\mathbb{E}|\zeta_n(t)-\zeta_n(s)|^\gamma \leq C_2|t-s|^{\frac{\gamma}{2}}.$$

• $t-s < \frac{1}{n}$. We have

$$\begin{aligned}
|\zeta_n(t)-\zeta_n(s)| &= |(W_n * k_n)(t) - (W_n * k_n)(s)| \\
&= | \ \int_{\mathbb{R}} [W_n(u)k_n(t-u) - W_n(u)k_n(s-u)]du| \\
&= | \ \int_{\mathbb{R}} W_n(u)[k_n(t-u) - k_n(s-u)]du| \\
&\leq \ \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{n}} (\sum_{i=1}^{n} |X_i| 1_{[\frac{i}{n},1[}(u) | [k_n(t-u) - k_n(s-u)]|)du
\end{aligned}$$

$$\leq \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{n}}(\sum_{i=1}^{n}|X_i| \; |\frac{1}{b_n}k(\frac{t-u}{b_n}) - \frac{1}{b_n}k(\frac{s-u}{b_n})|1_{[\frac{i}{n},1[}(u))du.$$

By inequality (11), we get

$$|\zeta_n(t)-\zeta_n(s)| \leq \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{n}}(\sum_{i=1}^{n}|X_i|\frac{1}{b_n}a(k)\frac{t-s}{b_n}|1_{[\frac{i}{n},1[}(u))du$$
$$\leq \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}|X_i|\frac{1}{b_n^2}a(k)|t-s|(1-\frac{i}{n})$$
$$\leq \frac{1}{\sigma\sqrt{n}} \quad \frac{1}{b_n^2}a(k) \sum_{i=1}^{n}|X_i||t-s|.$$

So

$$\frac{|\zeta_n(t)-\zeta_n(s)|}{|t-s|^\alpha} \leq \frac{1}{\sigma\sqrt{n}} \; \frac{1}{b_n^2}a(k) \sum_{i=1}^{n}|X_i| \; |t-s|^{1-\alpha}.$$

Thus

$$\omega_\alpha(\;\zeta_n,\frac{1}{n}) = \sup_{0<|t-s|<\frac{1}{n}} \frac{|\zeta_n(t)-\zeta_n(s)|}{|t-s|^\alpha}$$

$$\leq \frac{1}{\sigma\sqrt{n}} \quad \frac{1}{b_n^2}a(k)\sum_{i=1}^{n}|X_i|(\frac{1}{n})^{1-\alpha}$$

$$\leq a(k)\frac{1}{\sigma}\frac{1}{b_n^2}(\frac{1}{n})^{\frac{1}{2}-\alpha}(\frac{1}{n}\sum_{i=1}^{n}|X_i|).$$

To prove that $\omega_\alpha(\zeta_n,\frac{1}{n}) \xrightarrow{\mathbb{P}} 0$, it suffices to show that

$$a(k)\frac{1}{\sigma}\frac{1}{b_n^2} \; \frac{1}{n^{\frac{1}{2}-\alpha}}(\frac{1}{n}\sum_{i=1}^{n} \; |X_i|) \xrightarrow{P} 0.$$

The Markov inequality leads to

$$\mathbb{P}[(a(k)\frac{1}{\sigma}\frac{1}{b_n^2} \; \frac{1}{n^{\frac{1}{2}-\alpha}}(\frac{1}{n}\sum_{i=1}^{n}|X_i|)) > \lambda] \leq \frac{1}{\lambda}a(k)\frac{1}{\sigma}\frac{1}{b_n^2}\frac{1}{n^{\frac{1}{2}-\alpha}}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}|X_i|).$$

Hölder's inequality gives $\mathbb{E}|X_i| \leq (\mathbb{E}|X_i|^\gamma)^{\frac{1}{\gamma}} \leq M^{\frac{1}{\gamma}}$. Consequently

$$\mathbb{P}[(a(k)\frac{1}{\sigma}\frac{1}{b_n^2} \; \frac{1}{n^{\frac{1}{2}-\alpha}}(\frac{1}{n}\sum_{i=1}^{n}|X_i|)) > \lambda] \leq \frac{1}{\lambda}a(k)\frac{1}{\sigma} \; \frac{1}{b_n^2} \; \frac{1}{n^{\frac{1}{2}-\alpha}}(\frac{1}{n}\sum_{i=1}^{n}M^{\frac{1}{\gamma}})$$
$$\leq \frac{1}{\lambda}a(k)\frac{1}{\sigma}\frac{1}{b_n^2}\frac{1}{n^{\frac{1}{2}-\alpha}}(\frac{1}{n}nM^{\frac{1}{\gamma}})$$
$$\leq M^{\frac{1}{\gamma}}\frac{1}{\lambda}a(k)\frac{1}{\sigma}(\frac{1}{b_n^2}\frac{1}{n^{\frac{1}{2}-\alpha}}).$$

Since $M^{\frac{1}{\gamma}}\frac{1}{\lambda}a(k)\frac{1}{\sigma}(\frac{1}{b_n^2}\frac{1}{n^{\frac{1}{2}-\alpha}}) \underset{n\to\infty}{\to} 0$ for all $\alpha < \frac{1}{2}-\tau$. We obtain the tightness of $\{\zeta_n, n\geq 1\}$ in H_α^0, for all $\alpha < \min[\gamma^{-1}(\min(\gamma,\frac{\gamma}{2})-1),(\frac{1}{2}-\tau)] = \frac{1}{2}-\max(\frac{1}{\gamma},\tau)$. This completes the proof of the theorem.

References

[1] Ango-Nze, P., Doukhan, P. (2004), Weak dependence, models and applications to econometrics. *Econometric Theory*, **20(6)**, 995–1045.

[2] Bulinski, A., Shashkin, A. (2005), Strong invariance principle for dependent multi-indexed random variables. *Doklady Mathematics*, **72(1)**, 503–506.

[3] Bulinski, A., Shabanovich, E. (1998), Asymptotical behaviour of somme functionals of positively and negatively dependent random fields. *Fundman Appl. Math.*, **4**, 479–492.

[4] Dedecker, J., Doukhan, P. (2003), A new covariance inequality and applications. *Stoch. Proc. Appl.*, **106(1)**, 63–80.

[5] Dedecker, J. and al, (2007), Weak dependence: Models, Theory and Applications. *Lecture Note in Statistics 190, Springer-verlag.*

[6] Doukhan, P., Louhichi, S. (1999), A new weak dependence condition and application to moment inequalities. *Stoch. Proc. Appl.*, **84**, 313–342.

[7] Doukhan, P., Lang, G. (2002), Rates in the empirical central limit theorem for stationary weakly dependent random fields. *Stat. Inference Stoch. Proc.*, **5**, 199–228.

[8] Doukhan, P., Wintenberger, O. (2007), An invariance Principle For New Weakly Dependent Stationary general Models. *Probability And Mathematical Statistics*, **27**, Fasc. 1, 45–73.

[9] Doukhan, P., Newmann, M.H. (2007), Probability and Moment inequalities for sums of weakly dependent random variables, with applications. *Stoch. Proc. Appl.*, **117**, 878–903.

[10] Hamadouche, D. (2000), Invariance principles in Hölder spaces. *Portugaliae Mathematica*, **57**, 127–151.

[11] Kerkyacharian, G., Roynette, B. (1991), Une démonstration simple des théorèmes de Kolmogorov, Donsker et Ito-Nisio. *C. R. Acad. Sci. Paris Sér. Math. I*, **312**, 877–882.

[12] Lamperti, J. (1962), On convergence of stochastic processes. *Trans. Amer. Math. Soc.*, **104**, 430–435.

[13] Miamee, A.G., Pourahmadi, M. (1988), Wold decomposition and parametrization of stationary processes with infinite variance. *Probab. Th. Rel. Fields*, **79**, 145–164.

[14] Račkauskas, A., Suquet, Ch. (2004), Hölder norm test statistics for epidemic change. *Journal of Statistical Planning and Inference*, **126**, Issue 2, 495–520.