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Monodromy, hypergeometric polynomials and diophantine approximations

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Abstract

The monodromy's study of Fuchsian hypergeometric differential equation provides a natural framework for the explicit determination of rational approximations of polylogarithmic and hypergeometric functions. Thus, we can obtain almost without calculation explicit determination of many polynomials and hypergeometric power series related to their Padé approximations.
From now on, using a classical way, one can study the arithmetic nature of numbers related to the values taken by these functions.

Résumé

L'étude de la monodromie des equations différentielles hypergéométriques Fuchsiennes fournit un cadre naturel pour la détermination explicite des approximations rationnelles des fonctions polylogarithmes et de certaines fonctions hypergéométriques. Ceci permet de donner presque sans calcul certains polynômes et séries hypergéométriques intervenant dans la détermination des approximants de Padé liés à ces fonctions.
Par un cheminement désormais classique, on peut alors étudier la nature arithmétique de certains nombres en relation avec les valeurs prises par ces fonctions en certains points.

Keywords: Irrationality,linear independence over a field, generalized hypergeometric series, singularities, monodromy, local behavior of solutions,normal forms inverse problems (Riemann Hilbert inverse differential, etc).
Approximation by rational functions, Padé approximations.

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Contents

1 Introduction

♠ The aim of this article is to devote to a better understanding of many constructions of effective rational approximations to solutions of some linear hypergeometric functions from the perspective of the monodromy theory.
The monodromy's study of Fuchsian hypergeometric differential equation provides a natural framework for the explicit determination of rational approximations of polylogarithmic and hypergeometric functions.
Riemann [Rie] initiated this viewpoint in his study of Gauss's continued fraction expansion of

$$ {}_2F_1\left(\begin{array}{c} a+1, b \\ c+1 \end{array} \middle| x\right) / {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| x\right), \tag{1.1}$$

(see, also, [Hu1]).
More recently, G.V.Chudnovski [Chu2] has obtained many interesting results about linear forms involving generalized hypergeometric functions.
This, in turn, reveals a connection between Fuchsian differential equations and diophantine approximations of special Siegel G-functions.
♣ The arithmetic motivations of searching such effective rational approximations come for proving irrationality or transcendence of numbers arising as values of hypergeometric functions, such as

$$Li_q(1/p))\,, \zeta(2), \zeta(3), \cdots\ etc\,,$$

where : $Li_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^q}$ denotes the polylogarithmic function.
We recall that in 1978 R.Apery [Ap] proved that the number

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

was irrational.
Before Apery's result, nothing was known about the arithmetic properties of the zeta- functions at odd points.
The Apery 's constructions were generalized to the study of arithmetic of linear forms in odd zeta functions by Nesterenko, Ball-Rivoal and Zudilin in [Ne][Ne2],[Ba,Ri],[Zu1].
♣ Like in the explicit continued fraction expansions, we are interested in cases of the explicit determination of Padé approximations of the first case and the second case at infinity of a family

$$\mathcal{S} = \{f_1(z), f_2(z), \cdots, f_q(z)\}$$

where for $1 \leq k \leq q$, $f_k(z)$ are explicitely known by their power series expansions at $z = \infty$

$$f_k(z) = \sum_{n=1}^{\infty} a_n^{(k)} (1/z)^n \tag{1.2}$$

For instance , we shall take

$$f_k(z) = Li_k(1/z) = \sum_{n=1}^{\infty} \frac{1}{n^k}(1/z)^n$$

where Li_k denotes the polylogarihmic function.
♣ In this paper, we generalize results concerning the simultaneous rational approximations of these polylogarithmic functions given in [Hu1] and we effectively construct the system of Padé approximants of first and second kind at $z = \infty$ to $\mathcal{S}$ where for $1 \leq k \leq q$,

$$f_k(z) = \sum_{n=1}^{\infty} \frac{(1/z)^n}{(b_1+n)(b_2+n)\cdots(b_k+n)} \tag{1.3}$$

Let us remark that

$$f_{q-1}(z) = (\theta - b_q)(f_q(z)), \cdots, f_1(z) = (\theta - b_2)(f_2(z)) \tag{1.4}$$

where, as usual θ denotes the differential operator $\theta_z = z\frac{d}{dz}$.
In the following (for arithmetical reasons), for $1 \leq i \leq q$ we shall suppose that $b_i \in \mathbb{Q}, b_i \geq 0$.
We can check that $f_q(z)$ is a holomorphic solution at infinity of the following inhomogeneous Fuchsian differential equation of order $q+1$,(Euler's inhomogeneous equation).

$$(\theta - b_1)(\theta - b_2)\cdots(\theta - b_q)(f(z)) = \frac{(-1)^{q+1}}{1-z} \tag{1.5}$$

which is easily related to an hypergeometric differential equation.
This system of functions gives a very special contiguous system of local solutions at 0 of the previous Fuchsian differential equation.
Indeed $f_{q-1}(z), \cdots, f_1(z)$ are solutions of fuchsian differentials equations of order $q\,, q-1, \cdots 2$.
The polylogarithmic and the Lerch's functions are particular cases of these functions. (Take $b_1 = b_2 \cdots = b_q = 0$ resp $b_1 = b_2 \cdots = b_q = x$.)
It is easy to establish, by means of the standard - partial fraction decomposition that these functions $f_k(z)$ can be expressed by use of Lerch functions.
To obtain, almost without calculation explicit determination of many polynomials and hypergeometric power series related to their Padé approximations, we use the method of [Hu1].
We solve in this particular case the realization problem: given the subset, $S = \{0, 1, \infty\}$ of $\mathbb{P}_1(\mathbb{C})$ and given local numerical data of rank q on $Z =: \mathbb{P}_1(\mathbb{C})\backslash S$ we are asked to explicitly construct and analyze the behavior of Fuchsian differential equations that give rise to the given local system.
When the local system has no accessory parameters, that is, when it is "rigid" according to the terminology of Katz [Ka], the Fuchsian differential equation is unique and we obtain a Padé linear form as a special case of the contiguity

relations.
The rigidity of the local system or, equivalently the differential equation, has important arithmetic applications.
In particular, the uniqueness of the equation corresponding, respectively, to the generalized hypergeometric functions, the polylogarithmic and the Lerch functions

$$L_k(x,z) = \sum_{n=1}^{\infty} \frac{z^n}{(n+x)^k}$$

[Chu2], [Hu1], [Ne] enables us to establish many new explicit formulae for Padé approximations.
The polynomials discussed in this paper originate in the construction of simultaneous Padé approximation to these hypergeometric power series at the singular point $z = \infty$.
There are in particular hypergeometric and can also be obtained by the calculation of a multidimensional residues. There are often related to orthogonal polynomials.
Our general method does not so far directly reproduce or extend arithmetic results obtained by many cited authors and for the moment this paper may be view as a methodologic contribution.
In order to make our exposition self-contained, we shall first review in the next subsections some relevant notations and background material, omitting the proofs.

2 Padé approximants at infinity

♠ Let a family

$$\mathcal{S} = \{1, f_1(z), f_2(z), \cdots, f_q(z)\} \tag{2.1}$$

near the point $z = \infty$ on the Riemann sphere $\mathbb{P}_1(\mathbb{C})$ where these functions have "good" arithmetic expansions. Here, as in (1,2) for $1 \le k \le q$,$f_k(z)$ can be written

$$f_k(z) = \sum_{n=1}^{\infty} a_n^{(k)} (1/z)^n$$

There are two kinds of rational approximations than can be used.

Definition 1 ***Approximations of the first kind***
♣ *The first problem is to find polynomials*

$$\{A_0(z), A_1(z), \cdots, A_q(z)\} \tag{2.2}$$

such that $A_i(z)$, $1 \le i \le q$ *are of degrees at most* d_i *and such that expanding at* $z = \infty$ *we have :*

$$R(z) := A_0(z) + A_1(z) f_1(z) + \cdots A_q(z) f_q(z) \tag{2.3}$$

where

$$Ord_\infty R(z) := \sigma_\infty$$

is at least $\sigma = \sum_{k=1}^{q}(d_k + 1)$ *(which is theorically the best possible order and is equal to the number of the coefficients of* $A_k(z)$*.) It is called Hermite Padé approximant of the first kind (of weights* $(d_1, d_2, \cdots, d_q)$*).*

Uniqueness (up to a multiplicative constant) is not always ensured. It is sometimes useful to have an order σ_∞ which is less than σ. We note also that $deg\, A_0(z) \leq \min_{1\leq k\leq q} d_k - 1$.
The second one is called Padé approximants of the second kind .

Definition 2 ***Approximations of the second kind***
♣ *Let* $r_1, r_2, \cdots, r_q \geq 0$ *be integers . We put* $N = \sum_{j=1}^{q} r_j$*.*
The problem is to find a polynomial $Q_N(z)$ *not* $\equiv 0$ *of degree* N *such that there are polynomials* $P_{N,j}(z) \in \mathbb{C}[z]$ *of degrees at most* $N - 1$ *satisfying the interpolation conditions*

$$R_j(z) = Q_N(z)f_j(z) - P_{N,j}(z)$$

$Ord_\infty R_j(z) \geq r_j + 1$.
The polynomials $P_{N,j}(z)$ *are automatically determined as the polynomial parts of the corresponding series.*
This Padé approximant is called Padé approximant of second kind at infinity.

3 Differential hypergeometric equations

♠ In the following if $\alpha \in \mathbb{C}$ we put $(\alpha)_0 = 1$ and if $n \geq 1$,

$$(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2)\ldots(\alpha + n - 1)$$

(Pochhammer symbol)

$$_{q+1}F_q\left(\begin{array}{c} a_0, a_1, \cdots, a_q \\ b_1, b_2, \cdots, b_q \end{array}\middle| z\right) \tag{3.1}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{q}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!}$$

denotes the hypergeometric power series.
Provided that no denominator b_i is a non positive integer, the series coefficients are finite and the series converge absolutly in an open disk $|z| < 1$.
Moreover if

$$d = \Re(\sum_{k=1}^{q} b_k) - \Re(\sum_{k=1}^{q+1} a_k) > 0. \tag{3.2}$$

It converges on $|z| = 1$.
This power series is the holomorphic solution at 0 of the following differential equation of order $q+1$

$$\mathcal{H}yp((a)_i,(b)_i)$$

$$(\theta(\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_q-1)-z(\theta+a_0)(\theta+a_2)\cdots(\theta+a_q))y(z)=0. \quad (3.3)$$

where $\theta := z\frac{d}{dz}$.

The natural domain of definition of the solutions of the ordinary differential equation (ODE) is the Riemann -sphere $\mathbb{CP}_1$.
The ODE $\mathcal{H}yp((a)_i,(b)_i)$ has only $0,1,\infty$ as regular singular points and ${}_{q+1}F_q$ can be continued to a meromorphic function on $X = \mathbb{CP}_1 - \{0,1,\infty\}$, which is generally multivalued.
In fact, the solutions of $\mathcal{H}yp((a)_i,(b)_i)$ define a $(q+1)$-dimensional space of multivalued functions or **a local Fuchsian system.**
These functions are defined on the universal covering of X denoted by $\tilde{X}$.
To avoid multivaluedness $\mathbb{CP}_1$ is cut along $[1,\infty[$ and by définition ${}_{q+1}\mathcal{F}_q$ is the continuation of the power series from the disk $D(0,1)$ to $\tilde{X}$.
The solution space of any order ODE on $\mathbb{CP}_1$ is determined by the characteristic exponents.
To its $q+1-D$ space of solutions (comprising multivalued analytic functions on X) we associate a symbol called Riemann-P-scheme (see [In], [AAR]) which indicates the location of the singular points, and the exponents relative to each singularity.
The equation $\mathcal{H}yp((a)_i,(b)_i)$ is free of accessory parameters and the Riemann-P-scheme related to this equation is

$$P\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} \\ 0 & a_0 & 0 \\ 1-b_1 & a_1 & 1 \\ 1-b_2 & a_2 & 2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 1-b_q & a_q & d \end{pmatrix} |z \quad (3.4)$$

$f(z) =_{q+1} F_q(z)$ given by (3,1) is the unique power series which belongs to the zero exponent at $z=0$ and satisfies $f(0)=1$.
The sum of the $3(q+1)$ exponents equals $\frac{q(q+1)}{2}$ (Fuchs relation). So there are only $3q+2$ independent exponent parameters.
Recall that any $q+1$ Fuchsian ODE with three singularities is characterized by $3q+2$ independent exponents and $\frac{q(q-1)}{2}$ parameters, which together to the exponent parameters determine the global monodromy.
The $q+1$ exponents of the singularity $z=1$ are $0,1,2,\cdots q\,,d$ for some d, up to an overall additive constant, we say that this equation is of hypergeometric type.

The main point is that at $z=1$ there exists q holomorphic linearly independent solutions of $\mathcal{H}yp((a)_i,(b)_i)$.
This result is very important and is characteristic of the hypergeometric ODE.
When $d\in\mathbb{Z}$, one solution at $z=1$ is in general logarithmic $i.e$ can be written

$$\psi(z)=u(z)+(1-z)^d[v(z)\log(1-z)+w(z)] \tag{3.5}$$

where u is a polynomial of degree at most $q-1$ and v resp w are analytic functions at $z=1$.
Now, thanks to an fundamental study of Levelt [Le], which generalizes Riemanns , we can say that if an Fuchsian ODE , F of order $q+1$ which has singularities $0,1,\infty$ and no others, has in a neighborhood of the singularity 1 q linearly holomorphic independent solutions then, this ODE is an equation hypergeometric ODE $\mathcal{H}yp((a)_i,(b)_i)$.
The numbers $a_0,a_1\cdots a_q$, $0,1-b_1,\cdots,1-b_q$,$0,1,\cdots,q-1,d$ are called (Levelt) exponents at $z=0$,$z=\infty$,$z=1$.

3.1 Application: monodromy of the Lerch functions

♠ For $k\geq 2$, the power series $f_k(z)$ can be written

$$f_k(z)=\frac{(1+b_1)(1+b_2)\cdots(1+b_k)}{z}\,{}_{k+1}F_k\left(\begin{array}{c}1,b_1+1,\cdots,b_k+1\\ b_1+2,b_2+2,\cdots,b_k+2\end{array}\middle|1/z\right) \tag{3.6}$$

In the following, we suppose that $b_j-b_k\in\mathbb{Z}$.
As we can use partial- fraction decomposition for $f_k(z)$, it is sufficient to study the Lerch function.
This Lerch function (at $z=\infty$),

$$\Phi(x,k)(z)=\sum_{n=1}^{\infty}\frac{1}{(n+x)^k}(1/z)^n$$

is an holomorphic solution of the Euler's non-homogeneous equation

$$(\theta-x)^k(f(z))=\frac{(-1)^{k+1}}{1-z} \tag{3.7}$$

or the Fuchsian differential equation of order $k+1$

$$D(1-z)(\theta-x)^k)(f(z))=0$$

where D denotes as usual the operator $D=\frac{d}{dz}$.
When we compute the indicial equations at $z=0$, $z=\infty$,$z=1$, we obtain the following set of exponents :

- At $z=0$,$(0,x,x\cdots,x)$
- At $z=\infty$, $(1,-x,-x,\cdots,-x)$
- At $z=1$,$(0,1,2\cdots,k-1,k-1)$

One can conclude that this last exponent is related to a logarithmic solution as

in the formula (3,7).
A basis of solutions of the equation (1,4) at $z = \infty$ is

$$B(\Phi)_{x,k)} = (\Phi(x,k)(z), (1/z)^{-x}, (1/z)^{-x}\log(1/z), \cdots, (1/z)^{-x}(\log(1/z))^{k-1})$$

We use of analytic continuation of $\Phi_k(z), \cdots, \Phi_1(z)$ along loops γ_1 and γ_0 based in a vicinity of $z = 1$ resp $z = 0$
For instance the analytic continuation along γ_1 gives :

$$\Phi(x,k) \;\to\; \Phi(x,k)(z)|_{\gamma_1} = \Phi(x,k)(z) + 2i\pi\frac{\exp(-2i\pi x)}{(k-1)!}(\log(1/z))^{k-1} \quad (3.8)$$

This relation gives the monodromy's triangular matrix of size $(k+1)$, $M(x,k) = (a_{s,t})$ given by analytic continuation of the basis $B(\Phi)_{x,k}$ along the previous loops.
We note that

$$a_{k+1,k+1} = 2i\pi\frac{\exp(-2i\pi x)}{(k-1)!} \quad (3.9)$$

This monodromy's matrix is unipotent.

♣ Analytic continuation along the loops :

$$\gamma_1, \gamma_0 \circ \gamma_1 \cdots, \gamma_0^{q-2} \circ \gamma_1, \gamma_0^{k-1} \circ \gamma_1.$$

gives the following period's matrix (Deligne's period matrix related to mixed Hodge structure associated to this Lerch function,[DE]).

$$\Lambda(k,x) = \begin{pmatrix} 1 & \Phi(x,k)(z) & & \cdots\Phi(x,k)(z) \\ 0 & 2i\pi(1/z)^{(-x)} & & 2i\pi\log^{(q-1)}(1/z)/(q-1)!(1/z)^{-(x)} \\ & \ddots & & \\ 0 & \cdots 0 \cdots & (2i\pi)^{k-2}(1/z)^{(-x)} & (2i\pi)^{k-1}(1/z)^{(-x)}\log(1/z) \\ 0 & \cdots\cdots & 0 & (2i\pi)^{k}(1/z)^{(-x)} \end{pmatrix} \quad (3.10)$$

4 Local system and construction of Padé approximation

4.1 Padé approximation of the first kind

♠ We generalize results given by Valisenko. [Val]. In the following, we set $b_1 = b_2 = \cdots b_{k_1} = \beta_1$, $b_{k_1+1} = \cdots = b_{k_1+k_2} = \beta_2$,,$\cdots$,
$b_{k_1+\cdots+k_{m-1}+1} = \cdots = b_q = \beta_m$.
We suppose that

$$k_1 + k_2 + \cdots + k_m = q$$

We consider also the family of Lerch's functions,

$$\Phi_{b_s,k_t}(z) = \sum_{n=1}^{\infty} \frac{1}{(n+b_s)^{k_t}} (1/z)^n$$

Using partial fraction decomposition of $f_p(z)$, we can write

$$f_p(z) = \sum_{k=1}^{t} (\sum_{s=1}^{k_t} E_{k,t} \Phi_{b_s,k_t}(z))$$

Hence, the linear form

$$R_\infty(z) = B_0(z) + \sum_{k=1}^{q} B_k(z) f_k(z)$$

can be written

$$R_\infty(z) = A_0(z) + \sum_{j=1}^{m} (\sum_{s=1}^{k_s} A_{j,s}(z) \Phi_{b_j,k_s})$$

We now study this 'Lerch's " functions linear form.

4.2 Approximations of Lerch functions

♠ For an integer $q \geq 1$,x and z are complex numbers such $|z| \leq 1$,x is not a negative integer, we put

$$\Phi(x,q)(z) = \frac{1}{z(1+x)^q} \, {}_{q+1}F_q \left(\begin{array}{c} 1+x, 1+x \cdots, 1+x, 1 \\ 2+x, 2+x, \cdots, 2+x \end{array} \Big| 1/z \right) \tag{4.1}$$

Using the same method as [Hu1] we can find the Padé approximations of the family $\mathcal{S}$.

$$\mathcal{S} =: \mathbb{C}(z)\{\Phi(x,1)(z), \cdots, \Phi(x,q)(z)\}.$$

$\mathcal{S}$ is of rank $q+1$ over $\mathbb{C}(z)$.

Theorem 1 ♣ *The remainder $R_\infty(z)$ is a holomorphic solution of a Linear Fuchsian differential equation of order $q+1$.*
If we suppose that the characteristic exponents σ_1 and σ_0 relative to the analytic continuation at $z=1$ resp $z=0$ of $R_\infty(z)$ satisfy

$$\sigma_\infty + \sigma_0 + \sigma_1 = (q+1)n \tag{4.2}$$

then The remainder

$$R_\infty(z) = C_\infty(n)(1/z)^{\sigma_\infty} \, {}_{q+1}F_q \left(\begin{array}{c} \sigma_\infty + \sigma_0, \sigma_\infty + x, \sigma_\infty + x \\ \sigma_\infty + n + x + 1, \cdots, \sigma_\infty + n + x + 1 \end{array} \Big| 1/z \right) \tag{4.3}$$

Where the normalisation's constant is

$$C_\infty(n) = \frac{\Gamma(\sigma_\infty + x)^q (n!)^q}{\Gamma(\sigma_\infty + x + n + 1)^q} \frac{\Gamma(x - \sigma_0 + 1)\Gamma(x + \sigma_\infty)}{\Gamma(x + \sigma_\infty)}.$$

The polynomial $A_q(z)$ is hypergeometric namely :

$$A_q(z) =_{q+1} F_q \left(\begin{array}{c} -n, -n, \cdots, -n, \sigma_\infty + x \\ 1, 1, \cdots, 1, 1 + x - \sigma_0 \end{array} \middle| z \right) \tag{4.4}$$

The other polynomials are given by the Frobenius method of derivation with respect to a parameter t. One finds :

$$A_k(z) = \sum_{k=0}^{n} \frac{d^{q-k}(c_k(n+t))}{dt^{q-k}}|_{t=0} z^k.$$

The hypergeometric differential equation related to $R_\infty(z)$ is

$$(\theta - \sigma_0)(\theta - x)^q - z(\theta + \sigma_\infty)(\theta - n - x)^q = 0.$$

Proof

Let us recall the main steps of this proof which is almost the same as in ([Hu1]).
♣ We investigate the linear form

$$R_\infty(z) = \sum_{n=1}^{q} A_k(z)\Phi(k, x)(z) + A_0(z) \tag{4.5}$$

We put $Ord_\infty R_\infty(z) = \sigma_\infty$. $\mathcal{S}$ is of rank $q + 1$ over $\mathbb{C}(z)$.
We recall that the basic main idea uses Riemann's theorem about multivalued functions [Rie].
In the general case,we consider

$$\Omega = \mathbb{C} - \{0, 1, \infty, z_1, \cdots z_s\}$$

Let Γ_j be loops in the universal covering $\hat{\Omega}$ starting in a neighborhood of z_0 (say of $z = 0$) and surrounding the singular points $\{0, 1, \infty, z_1, \cdots z_s\}$

Theorem 2 (Riemann's theorem) *Let $g_1(z), g_2(z), \cdots g_{q+1}(z)$ be a system of multivalued and regular holomorphic functions on Ω that its Wronskian $det(g_i^{(j)} \neq 0$ and such that the prolongations of $g_j^\prime s$ along the loops Γ_j define automorphisms of the local system spanned by the $g_k^\prime s$.*
There exists an $q + 1 - th$ order differential equation with coefficients in $\mathbb{C}(z)$ of the fuchs type such that the system $g_1(z), g_2(z), \cdots g_{q+1}(z)$ of functions is its fundamental system.

♣ Now to find the Fuchsian linear differential equation of order $q + 1$ satisfied by $R_\infty(z)$ and analytic continuations of $R_\infty(z)$.
We consider an "adapted" basis of a local system.

For our problem, we use of analytic continuation of $\Phi_q(z), \cdots, \Phi_1(z)$ along loops γ_1 and γ_0 based in a vicinity of 1 resp 0.
This "adapted" basis will depend of the following loops :

$$\gamma_1, \gamma_0 \circ \gamma_1 \cdots, \gamma_0^{q-2} \circ \gamma_1, \gamma_0^{q-1} \circ \gamma_1.$$

to obtain the matrix of "periods" $\Lambda(k, x)$.
Now this form is related to the other linear forms obtained by use of analytic continuation of $R_\infty(z)$ by loops based in a vicinity of 1 resp ∞ (that is monodromy around the points 1 and ∞)

$$\gamma_1, \gamma_1 \circ \gamma_0 \cdots, \gamma_1 \circ \gamma_0^{q-2}, \gamma_1 \circ \gamma_0^{q-1}.$$

Using the same method as in [Hu1], we can find the exponents of this ODE.
One finds that the exponents are :
• At $z = \infty$,

$$(\sigma_\infty, -n + x, \cdots, -n + x).$$

• At $z = 0$ one finds

$$(\sigma_0, x, \cdots, x).$$

σ_0 is the exponent given by analytic continuation at $z = 0$ of $R_\infty(z)$. In general it is not an integer.
• At $z = 1$; $(0, 1, \cdots, q, \sigma_1)$
where σ_0) is the exponent given by analytic continuation at $z = 1$ of $R_\infty(z)$.
In this case, since $d = q$, σ_1 is an integer.
To obtain an hypergeometric differential equation , we suppose that these exponents given by analytic continuation of $R_\infty(z)$ satisfy

$$\sigma_0 + \sigma_\infty + \sigma_1 = (q + 1)n.$$

As all the exponents (which depend on the polynomials A_k) are determined within a nonnegative integer.
Now, we suppose that these exponents satisfy **Fuchs relation** :

$$\sigma_0 + \sigma_\infty + \sigma_1 - qn + \frac{q(q-1)}{2} = \frac{q(q+1)}{2}$$

or :

$$\sigma_0 + \sigma_\infty + \sigma_1 = (q + 1)n. \tag{4.6}$$

We can conclude that there do not exist apparent singularities.
This assumption is very important. If it is not satisfied, the differential equation satisfied by $R_\infty(z)$ has some apparent singularities and is very difficult to be computed.
This shows that the exponents are exacty those given by the previous study.
We thus obtain the hypergeometric Riemann scheme related to this problem

$$P\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & \sigma_\infty & 0 & \\ x & -n-x & 1 & \\ x & -n-x & 2 & |z \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ x & -n-x & \sigma_1 & \end{pmatrix} \tag{4.7}$$

♣ To find the Riemann scheme of the remainder, we consider the map $z \to 1/z$ in the previous Riemann scheme and consider,

$$(1/z)^{\sigma_\infty} P\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & \sigma_\infty+\sigma_0 & \sigma_1 & \\ -\sigma_\infty-x-n & \sigma_\infty+x & 0 & \\ -\sigma_\infty-x-n & \sigma_\infty+x & 1 & |1/z \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ -\sigma_\infty-x-n & \sigma_\infty+x & q-1 & \end{pmatrix} \tag{4.8}$$

This shows that the remainder $R_\infty(z)$ (within a multiplicative constant) is given by

$$R_\infty(z) = (1/z)^{\sigma_\infty}\ {}_{q+1}F_q\left(\begin{matrix} \sigma_\infty+x, \cdots, \sigma_\infty+x, \sigma_\infty+x, \sigma_\infty+\sigma_0 \\ \sigma_\infty+x+n+1, \cdots, \sigma_\infty+n+x+1 \end{matrix} \middle| 1/z \right) \tag{4.9}$$

The first quasi-polynomial $A_q(z)(1/z)^x$ satisfies also the same hypergeometric differential equation as $R_\infty(z)$.
We use of analytic continuation of $\Phi(q,x)(z), \cdots, \Phi_(1,x)(z)$ along loops γ_1 and γ_0 based in a vicinity of 1 resp 0.
We use the following loops :

$$\gamma_1, \gamma_0 \circ \gamma_1 \cdots, \gamma_0^{q-2} \circ \gamma_1, \gamma_0^{q-1} \circ \gamma_1.$$

to obtain the following relation (related to the matrix $\Lambda(q,x)$ and the root x of the indicial equation at $z=0$) :

If we denote by $\mathbb{B}(x)$ and $\mathbb{A}(x)$ the matrices of order $q+1$

$$\mathbb{B}(x) = (R_\infty(z), R_1(z), \cdots, R_q(z))^t$$

$$\mathbb{A}(x) = (A_0(z), A_1(z), \cdots, A_q(z))^t$$

where $()^t$ denotes the transpose of the matrix.
We find

$$\mathbb{B}(x) = \Lambda(q,x)\mathbb{A}(x) \tag{4.10}$$

As $R_q(z) = (2i\pi)^q(1/z)^x A_q(z)/(q-1)!$, it is now easy to give the polynomial $A_q(z)$,(up to a non zero multiplicative constant), one finds :

$$A_q(z) =_{q+1} F_q \left(\begin{array}{c} -n, -n, \cdots, -n, \sigma_\infty + x \\ 1, 1, \cdots, 1, 1 + x - \sigma_0 \end{array} \middle| z \right)$$

4.3 Multidimensional residues and computation of polynomials

In the previous section we have seen that the computation of the polynomial is given by the study of the differential hypergeometric equation. This computation gives polynomials up to a multiplicative constant.
The above study shows for instance that we can take as solution of the differential equation the particular integral :

$$R_\infty(z) = z^{\sigma_0} \int_0^1 \cdots \int_0^1 \frac{\prod_{k=1}^q t_k^{\sigma_\infty + x - 1}(1-t_k)^n}{(z - t_1 t_2 \cdots t_q)^{\sigma_\infty + \sigma_0}} dt_1 \cdots dt_q$$

We use this integral to compute the polynomial $A_q(z)$ and the other polynomials.
As analytic continuation along γ_1 gives :

$$R_\infty(z) \ \rightarrow \ R_\infty(z) + 2i\pi.Dz^x[(A_1(z) + A_2(z)(\log(1/z)) + \cdots A_q(z)\frac{(\log(1/z))^{q-1}}{(q-1)!}]$$

where $D = D(\sigma_\infty, \sigma_0)$ denotes a constant to be determined.

But this last expression depends of the deformation of the cycle γ_1 when this path cuts the segment $[0,1]$.
Suppose now that the point z crosses the cut $(1, +\infty[)$ in the plane of t_q,and describes a complete circuit about $zt_1 \cdots t_{q-1} = 1$.In the plane of t_q,the singular point $\tilde{t_q} = z/t_1 \cdots t_{q-1}$ describes a circuit in the same sense about $t_q = 1$.
The new form of the solution of the above integral solution is now found by deforming the contour of integration so that it never passes through a singular point of the integrand.
The segment $[0,1]$. becomes a path composed by a segment $[0, 1-\epsilon]$ composed with a line segment $[1-\epsilon, \tilde{t_q} - \epsilon]$, a small circle $C(\tilde{t_q}, \rho)$ and the line segment $[\tilde{t_q} + \epsilon, z]$, (Picard Lefchetz Principle).
We continue along $\gamma_0, \cdots \gamma_0^{q-1}$. We obtain an iterated integral and we follow by an integration along closed circles of radius ρ to obtain the value of $R_\infty(z)$.
A little computation gives :

$$D.A_q(z) = (-1)^{\sigma_0 + \sigma_\infty} z^{\sigma_0} \times$$

$$\prod_{k=1}^{q-1} \int_{|t_k|=\rho} t_k^{x-1-\sigma_0}(1-t_k)^n dt_1 \cdots dt_{q-1} \int_{|t_q - z/t_1 t_2 \cdots t_{q-1}|=\rho} \frac{t_q^{\sigma_\infty + x - 1}(1-t)^n}{(t_q - \frac{z}{t_1 \cdots t_{q-1}})^{\sigma_0 + \sigma_\infty}} dt_q$$

We put :

$$\int_{|t_q - z/t_1 t_2 \cdots t_{q-1}|=\rho} \frac{t_q^{\sigma_\infty + x - 1}(1-t)^n}{(t_q - \frac{z}{t_1 \cdots t_{q-1}})^{\sigma_0 + \sigma_\infty}} dt_q$$

then,

$$E = 2i\pi\, res(\frac{t_q^{\sigma_\infty + x-1}(1-t)^n}{(t_q - \frac{z}{t_1\cdots t_{q-1}})^{\sigma_0+\sigma_\infty}})|_{t_q=z/t_1t_2\cdots t_{q-1}}$$

$$= 2i\pi \frac{d^{\sigma_\infty+\sigma_0-1}}{dt_q^{\sigma_\infty+\sigma_0-1}}[t_q^{\sigma_\infty+x-1}(1-t_q)^n]|_{t_q=z/(t_1\cdots t_{q-1})}$$

Now, since

$$\int_{|t_k|=\rho} t_k^{-(k+1)}(1-t_k)^n dt_k = (-1)^k \binom{n}{k}$$

and

$$t_q^{\sigma_\infty+x-1}(1-t_q)^n = (\sum_{k=0}^{n}(-1)^k\binom{n}{k})t_q^{k+\sigma_\infty+x-1}$$

gives :

$$E = (2i\pi z^{x-\sigma_0}/(\sigma_\infty+\sigma_0-1)!)(\sum_{k=0}^{n}(-1)^{qk}(\binom{n}{k})^q A_k(z^k/(t_1\cdots t_{q-1})^{x+k-1})$$

where

$$A_k = \frac{\Gamma(x+\sigma_\infty)}{\Gamma(x-\sigma_0+1)\Gamma(x+\sigma_\infty)}\frac{(x+\sigma_\infty)_k}{(x+1-\sigma_0)_k}$$

Hence,this polynomial is obtained by the calculation of a multidimensional residue.
We find : $D = \frac{\Gamma(x+\sigma_\infty)}{\Gamma(x-\sigma_0+1)\Gamma(x+\sigma_\infty)}$ and

$$A_q(z) = {}_{q+1}F_q\left(\begin{array}{c} -n,-n,\cdots,-n,\sigma_\infty+x \\ 1,1,\cdots,1,1+x-\sigma_0\end{array}\middle| z\right)$$

Remark 1 *For $x = 1/2$ and $q = 2$,$z = -1$ we obtain simultaneous approximations of functions related to the Catalan's constant [RZ].*
If $\sigma_\infty = (q+1)n$, we are in the Padé's case.
If $1-n = 1-\sigma_0+x+\sigma_\infty+x$, i.e, we obtain the same result as [Riv].
Using the same method as [Hu1], we can obtain the well-poised and the very well -poised case. For instance , we obtain the same result as [Riv] with the same polynomial $A_q(z)$ and the same remainder $R_\infty(z)$.

4.4 The general case

We consider now the general linear form :

$$R_\infty(z) = A_0(z) + \sum_{j=1}^{m}(\sum_{s=1}^{k_s} A_{s,j}(z)f_{j,s})$$

where for $j = 1\cdots m$,$f_{j,s}(z) = \Phi(b_j,k_s))(z)$

Theorem 3 ♣ *The remainder $R_\infty(z)$ is a holomorphic solution of a Linear Fuchsian differential equation of order $q+1$.*
If we suppose that the characteristic exponents σ_1 and σ_0 relative to the analytic continuation at $z=1$ resp $z=0$ of $R_\infty(z)$ satisfy

$$\sigma_\infty + \sigma_0 + \sigma_1 = (q+1)n \tag{4.11}$$

then

$$R_\infty(z) = C_\infty(n)(1/z)^{\sigma_\infty}\ {}_{q+1}F_q\left(\begin{array}{c} \sigma_\infty + b_1, \cdots, \sigma_\infty + b_q, \sigma_\infty + \sigma_0 \\ \sigma_\infty + n + b_1 + 1, \cdots, \sigma_\infty + n + b_q + 1 \end{array} \middle| 1/z\right) \tag{4.12}$$

where the normalisation's constant is

$$C_\infty(n) = \frac{\prod_{j=1}^q \Gamma(\sigma_\infty + b_j)(n!)^q}{\prod_{j=1}^q \Gamma(\sigma_\infty + b_j + n + 1)}.$$

• *$R_\infty(z)$ can also be given by the following integral :*

$$R_\infty(z) = z^{\sigma_0} \int_0^1 \cdots \int_0^1 \frac{\prod_{j=1}^q t_j^{\sigma_\infty + b_j - 1}(1-t_j)^n}{(z - t_1 t_2 \cdots t_q)^{\sigma_\infty + \sigma_0}} dt_1 \cdots dt_q \tag{4.13}$$

or by the Mellin integral :

$$\frac{1}{2i\pi} \frac{\Gamma(b_1 + \sigma_\infty) \cdots \Gamma(b_q + \sigma_\infty)}{\Gamma(b_1 + n + \sigma_\infty + 1) \cdots \Gamma(b_q + \sigma_\infty + 1)\Gamma(\sigma_\infty + \sigma_0)} \tag{4.14}$$

$$\int_C \frac{\Gamma(b_1 + s + \sigma_\infty) \cdots \Gamma(b_q + s + \sigma_\infty)\Gamma(\sigma_\infty + s)}{\Gamma(b_1 + \sigma_\infty + s + n + 1) \cdots \Gamma(b_q + s + \sigma_\infty + n + 1)} . \Gamma(-s)(-z)^s ds$$

for $|\arg(-z)| < \pi$, where C is any path from $-i\infty$ to $i\infty$ such that the poles of $\Gamma(-s)$ lie on the right of C and the poles of $\Gamma(s + b_j$ lie on the left of C.
For $|\arg(-z)| < \pi$, where C is any path from $-i\infty$ to $i\infty$ such that the poles of $\Gamma(-s)$ lie on the right of C and the poles of $\Gamma(s + b_j)$ and $\Gamma(s + \sigma_\infty + \sigma_0)$ lie on the left of C.
♣ *If to simplify, we put p = k_s (multiplicity of the exponent b_m), the polynomial $A_{m,p}(z)$ related to $f_{m,k_s}(z)$ is the hypergeometric polynomial :*

$$A_{m,p}(z) =_{p+1} F_p\left(\begin{array}{c} -n, -n, -n \cdots, -n, \sigma_\infty + b_m \\ 1 + b_m - \sigma_0, 1, 1, \cdots, 1 \end{array} \middle| z\right) \tag{4.15}$$

Suppose now that b_m is a root of multiplicity j of the indicial equation at $z=\infty$. We put

$$A_{m,p}(z) = \sum_{j=0}^n c_j(n)\, z^j.$$

For $1 \le l \le j$, the polynomials $A_{m,p-l}(z)$ are given by :

$$A_{m,p-l}(z) = \sum_{k=0}^n \frac{d^l(c_j(n+t))}{dt^l}|_{t=0} z^k.$$

The hypergeometric differential equation related to $R_\infty(z)$ is

$$(\theta-\sigma_0)\Pi_{k=1}^{q}(\theta-b_k)-z(\theta+\sigma_\infty)\Pi_{k=1}^{q}(\theta-n-b_k)=0.$$

Proof

♣ Now as in the previous section, to find the Fuchsian linear differential equation of order $q+1$ satisfied by $R_\infty(z)$ and analytic continuations of $R_\infty(z)$,we have to find an "adapted" basis for the local system related to $R_\infty(z)$.
For our problem, we consider the analytic continuation of $f_{j,s}(z)$. along loops γ_1 and γ_0 based in a vicinity of 1 resp 0.
This "adapted" basis will depend of the following loops :

$$\gamma_1, \gamma_0\circ\gamma_1\cdots, \gamma_0^{q-2}\circ\gamma_1, \gamma_0^{q-1}\circ\gamma_1.$$

Using the same method as in [Hu1], we can find the exponents of this ODE.
One finds that the exponents are :
• At $z=\infty$,

$$(\sigma_\infty, -n+b_1, \cdots, -n+b_q).$$

• At $z=0$ one finds

$$(\sigma_0, b_1, \cdots, b_q).$$

σ_0 is the exponent given by analytic continuation at $z=0$ of $R_\infty(z)$. In general it is not an integer.
• At $z=1$; $(0,1,\cdots,q,\sigma_1)$
where σ_0) is the exponent given by analytic continuation at $z=1$ of $R_\infty(z)$.
In this case, since $d=q$, σ_1 is an integer.
Now, to obtain an differential hypergeometric equation, we suppose that these exponents given by analytic continuation of $R_\infty(z)$ (which depend on the polynomials $A_{k,s}$) are determined within a nonnegative integer satisfy **Fuchs relation** :

$$\sigma_0+\sigma_\infty+\sigma_1-qn+\frac{q(q-1)}{2}=\frac{q(q+1)}{2}$$

or :

$$\sigma_0+\sigma_\infty+\sigma_1=(q+1)n. \tag{4.16}$$

We can conclude that there do not exist apparent singularities.
We thus obtain the hypergeometric Riemann scheme related to this problem.

$$P\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & \sigma_\infty & 0 & \\ b_1 & -n-b_1 & 1 & \\ b_2 & -n-b_2 & 2 & |z \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ b_q & -n-b_q & \sigma_1 & \end{pmatrix} \tag{4.17}$$

♣ To find the Riemann scheme of the remainder, we consider the map $z \to 1/z$ in the previous Riemann scheme.

$$(1/z)^{\sigma_\infty} P \begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & \sigma_\infty + \sigma_0 & \sigma_1 & \\ -\sigma_\infty - b_1 - n & \sigma_\infty + b_1 & 0 & \\ -\sigma_\infty - b_2 - n & \sigma_\infty + b_2 & 1 & |1/z \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ -\sigma_\infty - b_q - n & \sigma_\infty + b_q & q-1 & \end{pmatrix} \tag{4.18}$$

Using (3,6) and (3,1) we can conclude that the remainder $R_\infty(z)$ (within a multiplicative constant) is given by

$$R_\infty(z) = (1/z)^{\sigma_\infty}\ {}_{q+1}F_q \left(\begin{matrix} \sigma_\infty + b_1, \cdots, \sigma_\infty + b_2, \sigma_\infty + b_q, \sigma_\infty + \sigma_0 \\ \sigma_\infty + b_1 + n + 1, \cdots, \sigma_\infty + n + b_q + 1 \end{matrix} \middle| 1/z \right) \tag{4.19}$$

If we choose the normalisation's constant of the remainder

$$C_\infty(n) = \frac{\prod_{k=1}^{q} (\Gamma(\sigma_\infty + b_k))^q \Gamma(\sigma_\infty + \sigma_0)}{\Gamma(\sigma_\infty + b_k + n + 1)^q (n!)^q},$$

we obtain the following integral formula :

$$R_\infty(z) = z^{\sigma_0} \int_0^1 \cdots \int_0^1 \frac{\prod_{k=1}^{q} t_k^{\sigma_\infty + b_k - 1} (1 - t_k)^n}{(z - t_1 t_2 \cdots t_q)^{\sigma_\infty + \sigma_0}} dt_1 \cdots dt_q$$

♣ Let us compute the polynomials $A_{m,p}(z)$.

To simplify the notations, we put $p = k_s$ (multiplicity of the exponent b_m) and compute the polynomial $A_{m,p}(z)$ related to $f_{m,k_s}(z)$.
As [Hu1],we consider the analytic continuation $R_{\gamma_1}(z)$ of $R_\infty(z)$ along the simple loop γ_1.
We use the symbol $diag[B_1, B_2, \cdots, B_m]$ for the diagonally blocked matrix B with diagonal block B_k, $B_k = B(\Phi)_{b_k}$ denoting a monodromy's matrix of size k_s .
Using this matrix and the relation (3,9), one sees that all the quasi-polynomial $z^{b_m} A_{m,p}(z)$ satisfy the same differential equation as $R_\infty(z)$ and in particular all the quasi- analytic solutions related to the exponent b_m and the Lerch's functions $\Phi(b_m, p)(z)$ are given by the following Riemann scheme of order p :

$$z^{b_m} P \begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 - b_m & \sigma_\infty + b_m & \sigma_1 & \\ 0 & -n & 0 & \\ 0 & -n & 1 & |z \\ \vdots & \vdots & \vdots & \\ 0 & -n & p-2 & \\ 0 & -n & p-1 & \end{pmatrix}.$$

The quasi -polynomials $A_{m,p}(z)$ which belongs to the " exponents "b_m ",(1 $\leq k \leq q$) are the same as in the formula (4,4) where one replaces x by b_m.
It can also be written (within a normalisation constant ,)

$$z^{b_m}\,{}_{q+1}F_q\left(\begin{matrix} -n, -n, \cdots, -n, -n+b_m \\ 1-\sigma_0+b_m, 1\cdots, 1 \end{matrix}\middle| z\right).$$

If,for instance b_m is a root of multiplicity j of the indicial equation at $z = \infty$, we use Frobenius method for solving Fuchsian differential equation [In] and if $1 \leq l \leq j$. We put

$$A_{m,p}(z) = \sum_{j=0}^{n} c_j(n)\, z^j.$$

For $1 \leq l \leq j$, the polynomials $A_{p-l}(z)$ are given by :

$$A_{p-l}(z) = \sum_{k=0}^{n} \frac{d^l(c_j(n+t))}{dt^l}|_{t=0} z^k$$

Now, as in the previous section, we can determine the polynomials given by the analytic continuations of the remainder.

Remark 2 *If we put*

$$P_n(b,t)) = \frac{1}{n!} t^{-b}(t^{b+n}(1-t)^n)^{(n)}$$

(Jacobi's polynomial),[Val].
After integrating by parts,we can show that for $\sigma_\infty = nq$ and $\sigma_0 = 0$,

$$R_\infty(z) = \int_0^1 \cdots \int_0^1 \frac{t_1^{\beta_1}\cdots t_j^{\beta_j} P_n(\beta_1,t_1)P_n(\beta_2+n,t_2)\cdots P_n(\beta_q+(q-1)n,t_q)}{(z-t_1t_2\cdots t_q)} dt_1\cdots dt_q$$

5 Padé approximation of the second kind

♠ Let $r_1 \geq r_2 \geq \cdots \geq r_m \geq 0$ be integers, and for each $i = 1, 2, \cdots m$ let k_i, the number of indices s for which $b_s = b_i$.
We put $N = \sum_{j=1}^{m} r_j$.
Let $b_1, b_2, \cdots, b_q$ be arbitrary nonnegative complex numbers, we give the Padé approximant of second kind at infinity for $f_1(z), \cdots, f_q(z)$.

Theorem 4 *We suppose that $\sigma_0 = \sigma_1 = 0$ then,*

$$Q_N(z) = \frac{(b_1+r_1+1)_{r_1}\cdots(b_q+r_q+1)_{r_q}}{r_1!r_2!\cdots r_q!} \tag{5.1}$$

$${}_{q+1}F_q\left(\begin{matrix} -N, r_1+b_1+1, \cdots, r_2+b_2+1, r_q+b_q+1 \\ b_1+1, b_2+1, \cdots, b_q+1 \end{matrix}\middle| z\right)$$

which can be written

$$Q_N(z) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \binom{r_1+k+b_1+1}{r_1} \cdots \binom{r_1+k+b_q+1}{r_q} z^k \tag{5.2}$$

Proof

♣ Using monodromy around $z = 1$ (analytic continuation along γ_1) i.e, we see that

$$Q_N(z), (1/z)^{-b_1} R_1(z), , (1/z)^{-b_2} R_2(z), \cdots (1/z)^{-b_q} R_q(z)$$

satisfy the same differential equation .

Using the same proof as the previous section, we can find the exponents at the singular points $z = 0$,$z = \infty$ and $z = 1$.

- At $z = \infty$ the exponents are $-N, b_1 + r_1 + 1, \cdots, b_q + r_q + 1$
- At $z = 0$, $\sigma_0, -b_1, \cdots, -b_q$.
- At $z = 1$, $\sigma_1, 0, 1, \cdots, q-1$.

If we suppose $\sigma_0 + \sigma_1 + \sum_{j=1}^{q} r_j - N + \frac{q(q-1)}{2} + q = \frac{q(q+1)}{2}$ (Fuchs relation). i.e

$$\sigma_0 + \sigma_1 + \sum_{j=1}^{q} r_j = N \tag{5.3}$$

We can conclude that there do not exist apparent singularities.

If $\sigma_0 = \sigma_1 = 0$, we obtain an hypergeometric differential equation and conclude that the Riemann scheme related to $Q_N(z)$ is

$$R \begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & -N & 0 & \\ -b_1 & r_1 + b_1 + 1 & 1 & \\ -b_2 & r_2 + b_2 + 1 & 2 & |z \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ -b_q & r_q + b_q + 1 & 0 & \end{pmatrix}. \tag{5.4}$$

The analytic solution related to this Riemann scheme is the polynomial $Q_N(z)$.

$$Q_N(z) =_{q+1} F_q \left(\begin{matrix} -N, r_1 + b_1 + 1, \cdots, r_q + b_q + 1 \\ 1 + b_1, 1 + b_2, \cdots, 1 + b_q \end{matrix} \; |z \right)$$

Now it is easy to find Hata's formulas concerning simultaneous rational approximations of Lerch's functions $\Phi(k, x)$, [Ha].

The polynomials P_j are given by

$$P_j(z) = \frac{1}{(r_j - 1)!} \int_0^1 \frac{Q_N(z) - Q_N(t)}{z - t} t^{-x} (\log(1/t))^{r_j} dt$$

and the remainders :

$$R_j(z) = \frac{1}{(r_j-1)!}\int_0^1 \frac{Q_N(t)}{z-t} t^{-x} (\log(1/t))^{r_j} dt$$

We can see that if we expand the left side of $R_j(z)$ in power of $1/z$ we have

$$ord_{z=\infty} R_j(z) \geq r_j + 1$$

We can also write :

$$Q_N(z) = \frac{1}{r_1!r_2!\cdots r_q!}((\prod_{k=1}^{q} z^{-x} D^{r_k} z^{x+r_k})(1-z)^N) \tag{5.5}$$

If $r_1 \geq r_2 \geq \cdots \geq r_q$, , one obtains simultaneous rational approximations of

$$1, L_1(1/z), \cdots Li_q(1/z)$$

(see [Hu1]) and [Zu3]) where we found the hypergeometric polynomial of degree N,

$$\begin{aligned} &{}_{q+1}F_q \left(\begin{matrix} -N, r_1+1, \cdots, r_2+1, r_q+1 \\ 1, 1, \cdots, 1 \end{matrix} \Big| z \right) \\ &= \sum_{k=0}^{n} \binom{N}{k}\binom{r_1+k}{k}\cdots\binom{r_q+k}{k}(-1)^k z^k \end{aligned}$$

If we consider the derivation's operator $D = \frac{d}{dz}$, since

$$D^k z^k = (\theta+1)\cdots(\theta+k),$$

we can write the polynomial $Q_N(z)$ as

$$Q_N(z) = \frac{1}{r_1!r_2!\cdots r_q!}\{\prod_{k=1}^{q}(\theta+1)(\theta+2)\cdots(\theta+r_k)\, {}_{q+1}F_q \left(\begin{matrix} -N, 1, 1, \cdots 1 \\ 1, 1, \cdots, 1, 1 \end{matrix} \Big| z \right). \tag{5.6}$$

and the relation

$${}_1F_0 \left(\begin{matrix} -N \\ - \end{matrix} \Big| z \right) =_{q+1} F_q \left(\begin{matrix} -N, 1, 1, \cdots 1 \\ 1, 1, \cdots, 1, 1 \end{matrix} \Big| z \right)$$

gives for this hypergeometric polynomial the formula :

$$Q_N(z) = \frac{1}{r_1!r_2!\cdots r_q!}(\prod_{k=1}^{q} D^{r_k} z^{r_k})(1-z)^N) \tag{5.7}$$

The differential hypergeometric equation satisfied by $Q_N(z)$ and the remainders $R_k(z)$ is

$$\theta^{q+1} - z(\theta - N)(\theta + r_1 + 1)\cdots(\theta + r_q + 1) = 0.$$

Since the exponents at the singular point $z = 0$ are all 0, the monodromy around $z = 0$ is maximally unipotent.
♣ The solutions of this differential equation are given by the use inverse-Mellin transform.
To find the polynomial' (given by analytic solution at 0 of this equation). (See [Iw]). One puts

$$f(z) = \frac{1}{2i\pi} \int_C g(t)(-z)^t dt.$$

We must find a condition on the function $g(t)$ under which this inverse Mellin transform converges and gives a solution of this differential equation (.Here C denotes a vertical line with possible deviation to avoid singularities of the integrand).
We find that $g(t)$ is a solution of the difference equation :

$$g(t+1) = -\frac{(t-N)(t+r_1+1)\cdots((t+r_q+1)}{(t+1)^{q+1}}.$$

We take a path of integration C as mentionned above.
Then if

$$\lim_{\tau \to \infty} (t^2 g(t)(-z)^t) = 0,$$

$(t = \sigma + i\tau)$ holds uniformly as $t \to \infty$ in any finite vertical trip, then the integral converges uniformly with respect to z and the above calculation can be legitimated.
In the following, we put $r_1 = r_2 = \cdots r_q = n$; $N = qn$ and we consider the solutions of this differential equation belonging to the exponent $n+1$ at infinity. Since $u = 1/z$ gives the transformation $\theta_u = -\theta_z$, we obtain the following differential equation :

$$(\theta_u + qn)(\theta_u - n - 1)^q - u(\theta_u)^{q+1} = 0$$

$g(t)$ satisfies

$$g(t+1) = -g(t)\frac{t^{q+1}}{(t+qn)(t-n-1)^q}$$

and we can take

$$g(t) = \frac{\Gamma(t)^{q+1}}{\Gamma(t+qn+1)\Gamma(t-n)^q}.$$

We simplify $g(t)$ and we find the rational function

$$R(t) = \frac{((t-1)(t-2)\cdots(t-n))^q}{t(t+1)\cdots(t+qn)}.$$

We now set $t = s + n + 1$ to find the integral form of the solutions for the previous differential equation

$$F(z) = \frac{\Gamma(s+n+1)^{q+1}}{\Gamma(s+1)^q \Gamma(s+(q+1)n+1)q!}.$$

$$\frac{1}{2i\pi}\int_{C-i\infty}^{C+i\infty}(R(s+n+1)(\frac{\pi}{\sin(\pi t)})^{q+1}(-1/z)^{s+n+1}ds$$

C is an arbitrary constant in the interval $0 < C < n+1$.
Using the residue's theorem we find :

$$F(z) = R_1(z) + R_2(z)\log(1/z) + \cdots R_q(z)\log(1/z)^q$$

This gives a sense to the substitution $z = 1$.
In particular, we obtain for $q = 2$ and for $z = -1$ simultaneous rational approximations of $\log 2, -\frac{\pi^2}{12}$ and, for $z = 1$, $q = 3$ simultaneous approximations for $\zeta(2)$ and $\zeta(3)$.

Remark 3 ♠ *As for the simultaneous approximations of*

$$L_z(1), Li_2(z), \cdots Li_q(z),$$

given in [Hu1] and [Hu2], we can use the method of perturbing power series for the construction of new differential equations to give the simultaneous rational approximations of the Lerch function.
In this case,we consider

$$Le_1(z,t) = \sum_{n=1}^{\infty}\frac{(1/z)^{n+t}}{(n+x+t)}$$

and for $k \geq 1$*,*

$$\mathcal{L}e_k(z) = \frac{1}{k!}\frac{\partial^k(Le_1(z,t))}{\partial t^k}|_{t=0}.$$

We obtain the same formulas as in the previous study.

5.1 The method of perturbing power series and the construction of new differential equations

♠ In [Zu3], Zudilin gives simultaneous approximations of

$$Li_1(1/z), Li_2(1/z), Li_3(1/z)$$

but where the numerator $P_j(z)$ are replaced by the rational function $\frac{P_j(z)}{(1-z)^n}$. resp $\frac{P_j(z)}{(1-z)^{2n}}$. Using our study, we can generalize his result.
♣ We put

$$Li_p(z,t) = \sum_{n=1}^{\infty}\frac{z^{n+t}}{(n+t)^p}$$

and for $k \geq 1$,

$$\mathcal{L}_p^k(z) = \frac{1}{k!}\frac{\partial^k(L_p(\frac{1}{z},t))}{\partial t^k}|_{t=0}.$$

We consider

$$\mathcal{R}(z) = \frac{1}{q!}\frac{\partial^q(R(z,t))}{\partial t^q}|_{t=0}$$

$$R(z,t) = Q(z)L_1(1/z,t) - \frac{P(z,t)(\frac{1}{z})^t}{(1-z)^{(q-1)n+1}}$$

where $Q(z) = (1-z)^{q(n-1)}Q_n(z)$,$Q_n(z)$ being a polynomial of degree n and $P(z,t) \in \mathbb{C}(t)[z]$ is of degree $qn-1$ in z.

$$R(z) = \sum_{j=1}^{q}(-1)^j \frac{\binom{k}{j}}{(q-j)!}[Q(z)Li_{1+j}(1/z) - \frac{\partial^j P(z,t)}{\partial t^j}|_{t=0}](\log 1/z)^{q-j}) \quad (5.8)$$

but if we write $R(z,t) = S(z,t)(1/z)^t$ where

$$S(z,t) = \sum_{n=\sigma_\infty}^{\infty} a_n(t)(1/z)^n$$

and $a_n(t) \in \mathbb{C}(t)$, then

$$R(z) = \frac{1}{q!}(\sum_{j=0}^{q}\binom{q}{j}(\frac{\partial^j S(z,t)}{\partial t^j}|_{t=0})(\log 1/z)^{q-j}) \quad (5.9)$$

♣ We obtain simultaneous rational approximations of

$$Li_1(1/z), Li_2(1/z), \cdots, Li_q(1/z).$$

The number of coefficients of the polynomial $Q_n(z)$ is now $n+1$.
The exponents related to $R(z)$ are now :
• At $z = \infty$ are $(-qn, n+1, \cdots n+1)$.
• At $z = 0$, $(\sigma_0, 0, \cdots, 0)$.
• At $z = 1$,$(q(n-1), q(n-1)+1, \cdots, q(n-1)+q-1, \sigma_1)$
We consider now the differential equation satisfied by $\tilde{R}(z) = (1-z)^{-q(n-1)}R(z)$.
the exponents related to $\tilde{R}(z)$ are :
• At $z = \infty$ are $(-n, n+1, \cdots n+1)$.
• At $z = 0$, $(\sigma_0, 0, \cdots, 0)$.
• At $z = 1$,$(0, 1, \cdots, q-1, \sigma_1)$
To obtain an hypergeometric differential equation, the following relation must be satisfied

$$\sigma_0 - n + q(n+1) + \sigma_1 + \frac{q(q-1)}{2} = \frac{q(q+1)}{2}$$

i.e. $\sigma_0 + \sigma_1 - q(n-1) = 0$.(Fuchs relation)
If, for instance we choose $\sigma_0 = 0$ and $\sigma_1 = -(q-1)n$, this condition is satisfied.

We can deduce that : $\mathcal{R}(z)$ and $Q_n(z)$ have the following Riemann scheme

$$R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & -n & 0 & \\ 0 & n+1 & 1 & \\ 0 & n+1 & 2 & \\ \vdots & \vdots & \vdots & |z \\ \vdots & \vdots & \vdots & \\ 0 & n+1 & -(q-1)n & \end{pmatrix} \tag{5.10}$$

We then find

$$Q_n(z) = {}_{q+1}F_q\left(\begin{matrix} -n, n+1, n+1, \cdots n+1 \\ 1, 1, \cdots, 1, 1 \end{matrix} |z \right).$$

or

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{n}^q z^k \tag{5.11}$$

For $q = 2$ and $z = 1$ thanks to Thomae's transformation of ${}_3F_2(1)$-hypergeometric series, namely,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{n}^2 = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \binom{n+k}{n}$$

Zudilin in [Zu3] has remarked that the latter sum gives (up to the sign factor) the denominators of Apéry's approximations of $\zeta(2)$.
We can use the same method to find the rational function related to the remainder given in [Zu3]

$$R(t) = \frac{((t-1)(t-2)\cdots(t-n))^q}{n!(t+1)(t+2)\cdots(t+qn)}.$$

The Barnes integral that we consider is

$$\frac{1}{2i\pi}\int_{C-i\infty}^{C+i\infty} R(t+n+1)(\frac{\pi}{\sin(\pi t)})^{q+1}(-1/z)^{t+n+1}dt$$

C is an arbitrary constant in the interval $0 < C < n+1$. (We can take $C = -1/2$).
This result coincides with Zudilin'result for $q = 2$ and $q = 3$.
But as in [Zu3], no new irrationality and linear independence results are presented. We just tried to give some sense to certain hypergeometric series that are expressed in terms in polylogarithms and are divergent when one formally plugs z with $|z| = 1$.
The novelty is the use of differential equation to give a systematic procedure to give such simultaneous rational approximations of polylogarithmic or Lerch's functions.

We use the same construction to find the Apery's differential equation.

5.2 Differential equations and simultaneous approximations of $\zeta(2)$ and $\zeta(3)$

♠ For $p = 2$, $k = 1$, we obtain the Apery's case. We use the linear form

$$R(z) = \frac{\partial}{\partial t}((A_2)(z)Li_2(1/z,t) + A_1(z)Li_1(1/z,t) + A_0(z,t)(\frac{1}{z^t})|_{t=0}$$

where $A_0(z,t) \in \mathbb{C}(t)[z]$.
The differential operator related to this function is (for more details see [Hu1]). In this case, using the previous study it is not difficult to show that the Riemann scheme is given by: (see also [Gu])

$$R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & -n & 0 & \\ 0 & -n & 1 & |z \\ 0 & n+1 & 2 & \\ 0 & n+1 & 1 & \end{pmatrix}. \tag{5.12}$$

It is related to the hypergeometric differential equation

$$\theta^4 - z(\theta - n)^2(\theta + n + 1)^2 = 0. \tag{5.13}$$

The monodromy around $z = 0$ is maximally unipotent ! the Apéry's polynomial [Ap] is given by

$$A_3(z) = {}_4F_3\left(\begin{matrix} -n, -n, n+1, n+1 \\ 1, 1, 1 \end{matrix} |z\right). \tag{5.14}$$

This differential equation is also unipotent.
The previous Riemann scheme can be written :

$$\frac{1}{z^{n+1}}R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & n+1 & 0 & \\ 0 & n+1 & 1 & |1/z \\ -2n-1 & n+1 & 2 & \\ -2n-1 & n+1 & 1 & \end{pmatrix}. \tag{5.15}$$

The nonlogarithmic solution is thus given by :

$$\frac{1}{z^{n+1}}\,{}_4F_3\left(\begin{matrix} n+1, n+1, n+1, n+1 \\ 1, 2n+2, 2n+2 \end{matrix} |\frac{1}{z}\right) \tag{5.16}$$

(within a multiplicative constant).
If one puts, $R_1(z) = (\frac{(n!^4}{((2n+1)!)^2}\frac{1}{z^{n+1}})r_1(z)$ and denoting by

$$r_1(z) = \sum_{n=0}^{\infty} c_n(1/z)^n,$$

the logarithmic function is given by $r_2(z) = \frac{\partial}{\partial t}(\sum_{k=0}^{\infty} c_{n+t}(1/z)^{n+t})|_{t=0}$.
♣ Let us recall the sketch of the method given in [Gu] and [Ne] to find the solutions of this differential equation.
The 'Apery's polynomial' are given by the use of inverse-Mellin transform (given by analytic solution at 0 of (6,3). Setting

$$f(z) = \frac{1}{2i\pi} \int_C g(t)(-z)^t dt,$$

we find a condition on the function $g(t)$ under which this inverse Mellin transform converges and gives a solution of this differential equation. Here C denotes a vertical line with possible deviation to avoid singularities of the integrand.
We find that $g(t)$ is a solution of the difference equation

$$g(t+1) = -g(t)\frac{(t-n)^2(t+n+1)^2}{(t+1)^4}$$

We take a path of integration C as mentionned above.
If

$$\lim_{\tau \to \infty} (t^2 g(t)(-z)^t) = 0,$$

($t = \sigma + i\tau$) holds uniformly as $t \to \infty$ in any finite vertical trip, then the integral converges uniformly with respect to z.
For instance with

$$g(t) = \frac{\Gamma(-t)\Gamma(1+n+t)^2}{\Gamma(1+t)^3\Gamma(1+n-t)^2}$$

after simplification, we find:

$$f(z) = \frac{1}{2i\pi} \int_C R(t)(-z)^t dt$$

$$R(t) = [\frac{(t-1)(t-2)\cdots(t-n)}{(t)(t+1)\cdots(t+n)}]^2.$$

To find the remainder (*i.e.*) the solution belonging to the exponent $n+1$ at ∞, we set $u = 1/z$, the new equation becomes

$$u\theta_u^4 - (\theta_u + n)^2(\theta_u - n - 1)^2 = 0$$

and

$$g(u) = \frac{\Gamma(u)^4}{\Gamma(u+n+1)^2\Gamma(u-n)^2} = \frac{\Gamma(u)^4\Gamma(1+n-u)^2}{\Gamma(u+n+1)^2}(\frac{\pi}{\sin(\pi u)})^2.$$

We can write $f(u)$ as

$$f(u) = \frac{1}{2i\pi} \int_C g(t)(-u)^t dt$$

or

$$f(u) = \frac{1}{2i\pi} \int_C R(t)(\frac{\pi}{\sin(\pi t)})^2(-u)^t dt$$

C is the vertical line $\Re u = C$, $0 < C < n+1$, oriented from top to the bottom.
(See [Ne] for the end of the proof).

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