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Functional central limit theorems for self-normalized partial sums of linear processes

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Abstract

We prove the invariance principle under self-normalization by blocks for linear processes with summable filters and i.i.d. innovations in the domain of attraction of the normal distribution.

Résumé

Nous prouvons un principe d'invaraince avec autonormalisation par blocs pour des processus linéaires à filtre sommable et innovations i.i.d. dans le domaine d'attraction de la loi normale.

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1 Introduction and results

We consider linear processes

$$X_k = \sum_{i\in\mathbb{Z}} a_i \epsilon_{k-i}, \quad k \in \mathbb{N}, \tag{1}$$

where $(a_i, i \in \mathbb{N})$ is a square summable ($\sum_{i\in\mathbb{Z}} a_i^2 < \infty$) sequence of real numbers and $(\epsilon_i, i \in \mathbb{Z})$ are i.i.d. centered random variables in the domain of attraction of the normal law (writen $\epsilon_1 \in \mathrm{DAN}$). This implies in particular that $\mathbf{E}\,|\epsilon_i|^p < \infty$ for each $0 < p < 2$ and, consequently X_k is well defined (see, e.g., Brockwell and Davis [4]). Central limit theorem for partial sums $S_n = X_1 + \cdots + X_n, n \in \mathbb{N}$, and functional limit theorems for processes build from partial sums $(S_k, k \in \mathbb{N})$ has been extensively studied in the literature. We refer to the survey paper by Merlevède, Peligrad and Utev [15] for recent results on the central limit theorem and its weak invariance principle for stationary sequences under finite second moment assumption. In this paper we shall not assume that ϵ_i has finite variance. In such context self-normalization is an appropriate technique. Usually this means the normalization by V_n, where

$$V_n^2 = X_1^2 + \cdots + X_n^2. \tag{2}$$

A rich literature is devoted to the limit behavior of the sequence $V_n^{-1} S_n$ in the case of independent random variables $X_1, X_2, \ldots$ Particularly, the central limit theorem is completely solved by Giné, Götze, and Mason [8] and invariance principles were proved in Račkauskas and Suquet [18, 19], Csörgő, Szyszkowicz and Wang [5]. For other important aspects of limit behaviour of self-normalized sums we refer to Logan et al. [14], Hahn and Zhang [9] and references therein. For self-normalization techniques in time series analysis, we refer to Klüppelberg and Mikosch [12]. We also refer to a survey paper by de la Peña, Klass and Lai [6] for recent results of the theory and applications of self-normalized processes in dependent variables.

We consider self-normalization using block-sums $B_{m,j}$ of X_k's. Choosing $N = [n/m]$, where $m = m(n)$, define

$$U_n^2 = \sum_{j=1}^{N} B_{m,j}^2, \quad \text{with} \quad B_{m,j} = \sum_{i=(j-1)m+1}^{jm} X_i. \tag{3}$$

Let us remind that $\epsilon_1 \in \mathrm{DAN}$ (domain of attraction of the normal distribution) means that there exists a sequence $b_n \uparrow \infty$ such that

$$b_n^{-1} \sum_{k=1}^{n} \epsilon_k \xrightarrow[n\to\infty]{} \mathfrak{N}(0,1), \quad \text{in distribution.} \tag{4}$$

The following theorem is our contribution to the central limit theorems for linear processes.

Theorem 1. *If* $\sum_{i\in\mathbb{Z}}|a_i| < \infty$, $\sum_{i\in\mathbb{Z}} a_i \neq 0$ *and* $\epsilon_1 \in \mathrm{DAN}$, *then*

$$U_n^{-1} S_n \xrightarrow[n\to\infty]{} \mathfrak{N}(0,1), \quad \textit{in distribution}, \tag{5}$$

provided that $m \to \infty$ *and* $m/n \to 0$ *as* $n \to \infty$.

Clearly the condition "$\epsilon_1 \in \mathrm{DAN}$" is necessary for the convergence (5) to hold on the whole class of filters considered in Theorem 1. Indeed looking at the special case where $a_0 = 1$ and $a_i = 0$ for $i \neq 0$, we have $X_k = \epsilon_k$ and then the membership of ϵ_1 in DAN is necessary by Giné, Götze, and Mason [8].

Earlier the convergence (5) was obtained by Juodis and Račkauskas [10] under the stronger condition $\sum_j j|a_j| < \infty$. Theorem 1 is actually a corollary of functional limit theorems proved in this paper.

We define the polygonal line process

$$\xi_n(t) = \sum_{i=1}^{[nt]} X_i + (nt - [nt]) X_{[nt]+1}, \quad t \in [0,1] \tag{6}$$

and view it as a random element in the Banach space $C[0,1]$ of continuous functions on $[0,1]$ equipped with the uniform norm

$$||x|| = \sup_{t\in[0,1]} |x(t)|. \tag{7}$$

We consider also the step partial sums process $\{\zeta_n(t) : t \in [0,1]\}$ defined by

$$\zeta_n(t) = \sum_{i=1}^{[nt]} X_i \tag{8}$$

as a random element of the Skorohod space D[0, 1] of all functions on [0, 1] which have left-hand limits and are continuous from the right, equipped with the Skorohod topology (see, e.g. [2, Section 14].

Let $W = \{W(t) : t \in [0,1]\}$ denote the standard Brownian motion on $[0,1]$. By $\xrightarrow{\mathcal{D}}$ we denote the convergence in distribution in the indicated space.

Our main results are the following two theorems.

Theorem 2. *If* $\sum_{i\in\mathbb{Z}}|a_i| < \infty$, $\sum_{i\in\mathbb{Z}} a_i \neq 0$ *and* $\epsilon_1 \in \mathrm{DAN}$, *then*

$$U_n^{-1}\xi_n \xrightarrow{\mathcal{D}} W \quad \textit{in the space} \quad C[0,1], \tag{9}$$

as $n \to \infty$, *provided that* $m \to \infty$ *and* $m/n \to 0$ *as* $n \to \infty$.

Theorem 3. *If* $\sum_{i\in\mathbb{Z}}|a_i| < \infty$, $\sum_{i\in\mathbb{Z}} a_i \neq 0$ *and* $\epsilon_1 \in \mathrm{DAN}$, *then*

$$U_n^{-1}\zeta_n \xrightarrow{\mathcal{D}} W \quad \textit{in the space} \quad D[0,1], \tag{10}$$

as $n \to \infty$, *provided that* $m \to \infty$ *and* $m/n \to 0$ *as* $n \to \infty$.

Using the same assumption about the a_i's, Kulik [13] obtained a strong approximation result for the process $\beta^{-1}V_n^{-1}\zeta_n$ with $\beta = |\sum_i a_i| \, (\sum_i a_i^2)^{-1/2}$. In view of statistical applications, the interest of the self-normalization by U_n instead of V_n is that we do not need to know this coefficient β. Of course in practical situations, the choice of $m = m(n)$ is an important problem. Obtaining an optimal choice under so general assumptions as above seems out of reach.

2 Useful facts

We gather here some information about the DAN property, selfnormalization and linear processes, used in the proofs of our limit theorems. To avoid notational confusion, we denote by $V_n(\epsilon)$, $U_n(\epsilon)$, $\zeta_n^{(\epsilon)}$, $\xi_n^{(\epsilon)}$ the objects defined substituting X by ϵ in (2), (3), (8), (6) respectively.

If $\epsilon_1 \in \mathrm{DAN}$, then with the normalizing sequence (b_n) as in (4), one has for each $\tau > 0$,

$$nP(|\epsilon_1| > \tau b_n) \xrightarrow[n\to\infty]{} 0, \tag{11}$$

$$\frac{n}{b_n^2}\mathbf{E}\,\epsilon_1^2\mathbf{1}_{\{|\epsilon_1|\leq\tau b_n\}} \xrightarrow[n\to\infty]{} 1, \tag{12}$$

see e.g. Araujo and Giné [1], Chap. 2, Cor. 4.8(a), Th. 6.17 (i) and Cor. 6.18 (b). Moreover

$$b_n^{-2}\sum_{k=1}^{n}\epsilon_k^2 \xrightarrow[n\to\infty]{\mathrm{Pr}} 1. \tag{13}$$

Lemma 4. *If $\epsilon_1 \in \mathrm{DAN}$ with normalizing sequence $(b_n)_{n\geq 1}$ as in (4), then*

$$\frac{n}{b_n}\mathbf{E}\,\left(|\epsilon_1|\,\mathbf{1}_{\{|\epsilon_1|>b_n\}}\right) \xrightarrow[n\to\infty]{} 0. \tag{14}$$

Proof. Integrating by part we obtain

$$\frac{n}{b_n}\mathbf{E}\,\left(|\epsilon_1|\,\mathbf{1}_{\{|\epsilon_1|>b_n\}}\right) = nP(|\epsilon_1| > b_n) + \frac{n}{b_n}\int_{b_n}^{\infty} P(|\epsilon_1| > x)\,\mathrm{d}x.$$

The first term in the above right hand side tends to zero by (11). To prove the same convergence for the second term, it is convenient to express $P(|\epsilon_1| > x)$ in terms of the truncated second moment $L(x) := \mathbf{E}\,\epsilon_1^2\mathbf{1}_{\{|\epsilon_1|\leq x\}}$. Since $\epsilon_1 \in \mathrm{DAN}$, it is well known (e.g. Feller [7]) that L is slowly varying and that $x^2P(|\epsilon_1| > x) = o(L(x))$ as $x \to \infty$. This may be rewritten as

$$P(|\epsilon_1| > x) = \frac{L(x)g(x)}{x^2}, \quad x > 0, \tag{15}$$

with some non negative function g such that $g(x)$ tends to 0 at infinity. In particular, g is bounded on some interval $[c, \infty)$. Now by a Karamata result (see [3, Th. 1.5.3] or [11, pp.45–46]), for every $\delta > 0$, $x^{-\delta}L(x)$ is asymptotic to a non increasing function. Fixing $\delta \in (0,1)$, there exists then some n_0 such that for every $n \geq n_0$, $b_n \geq c$ and

$$\frac{L(x)}{x^{\delta}} \leq \frac{2L(b_n)}{b_n^{\delta}}, \quad x \geq b_n. \tag{16}$$

Using (15) and (16), we obtain

$$\frac{n}{b_n}\int_{b_n}^{\infty} P(|\epsilon_1| > x)\,\mathrm{d}x \leq \frac{2nL(b_n)}{b_n^{1+\delta}}\int_{b_n}^{\infty}\frac{g(x)}{x^{2-\delta}}\,\mathrm{d}x = \frac{2nL(b_n)}{b_n^2}\int_1^{\infty}\frac{g(b_n t)}{t^{2-\delta}}\,\mathrm{d}t.$$

This last estimate tends to 0 applying (12) and the bounded convergence theorem (recall that g is bounded on $[c,\infty)$). □

Lemma 5 (see Lemma 9 in [19]). *If* $\epsilon_1 \in \mathrm{DAN}$ *then*

$$\sup_{t\in[0,1]}\left|\frac{V_{[nt]}^2(\epsilon)}{V_n^2(\epsilon)} - t\right| \xrightarrow[n\to\infty]{\Pr} 0.$$

Theorem 6 (see Th. 2.1 in [18]). *The convergence*

$$V_n^{-1}(\epsilon)\xi_n^{(\epsilon)} \xrightarrow[n\to\infty]{\mathcal{D}} W \quad \text{in the space } C[0,1]$$

holds if and only if $\epsilon_1 \in \mathrm{DAN}$.

Lemma 7 (see Lemma 4 in [10]). *If* $\epsilon_1 \in \mathrm{DAN}$ *then*

$$\frac{U_n^2(\epsilon)}{V_n^2(\epsilon)} \xrightarrow[n\to\infty]{\Pr} 1.$$

The following key lemma is essentially an adaptation of Lemma 1 of Peligrad and Utev [16]. It plays a key role to transfer some limit theorems from the innovations to the linear process.

Lemma 8 (see e.g. Lemma 1 in [17]). *Let* $(a_i)_{i\in\mathbb{Z}}$ *be a collection of real numbers, satisfying*

$$\sum_{i\in\mathbb{Z}} |a_i| < \infty. \tag{17}$$

Assume that $(Z_{n,i},\ n\in\mathbb{N}, i\in\mathbb{Z})$ *is a collection of random elements with values in a separable Banach space* $(\mathbb{E}, ||\cdot||_{\mathbb{E}})$ *satisfying the following conditions:*

(*i*) $\sup_{n\in\mathbb{N},i\in\mathbb{Z}} \mathbf{E}\,||Z_{n,i}||_{\mathbb{E}} < \infty$,

(*ii*) *For every fixed* $i,j\in\mathbb{Z}$ *it holds* $||Z_{n,i} - Z_{n,j}||_{\mathbb{E}} \xrightarrow[n\to\infty]{\Pr} 0.$

Then for each $n\in\mathbb{N}$ *the series* $\sum_{i\in\mathbb{Z}} a_i Z_{n,i}$ *converge a.s. and for every index* $\ell\in\mathbb{Z}$, *the following convergence*

$$\Big\|\sum_{i\in\mathbb{Z}} a_i Z_{n,i} - A Z_{n,\ell}\Big\|_{\mathbb{E}} \xrightarrow[n\to\infty]{\Pr} 0 \tag{18}$$

holds, where $A = \sum_{i\in\mathbb{Z}} a_i$.

We state here for reference convenience the following special case of Lemma 8 where $\mathbb{E} = \ell^2(\mathbb{N})$ and $Z_{n,i} = \big(Z_{n,i}^{(j)}\big)_{j\in\mathbb{N}}$.

Lemma 9. *Let $(a_i)_{i\in\mathbb{Z}}$ satisfies (17). Assume that $(Z_{n,i}^{(j)}, n, j \in \mathbb{N}, i \in \mathbb{Z})$ is a collection of random variables satisfying the following conditions:*

(i) $\sup_{n\in\mathbb{N},\ i\in\mathbb{Z}} \mathbf{E}\Big(\sum_{j\in\mathbb{N}}[Z_{n,i}^{(j)}]^2\Big)^{1/2} < \infty$;

(ii) for any $i, k \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}}(Z_{n,i}^{(j)} - Z_{n,k}^{(j)})^2 \xrightarrow[n\to\infty]{\mathrm{Pr}} 0$.

Then with any $k \in \mathbb{Z}$ it holds

$$\Delta_n := \Big|\Big(\sum_{j\in\mathbb{N}}\Big(\sum_{i\in\mathbb{Z}} a_i Z_{n,i}^{(j)}\Big)^2\Big)^{1/2} - \Big(A^2\sum_{j\in\mathbb{N}}[Z_{n,k}^{(j)}]^2\Big)^{1/2}\Big| \xrightarrow[n\to\infty]{\mathrm{Pr}} 0.$$

3 Proof of the limit theorems

Theorem 1 is an immediate consequence of Theorem 2 since $S_n = \xi_n(1)$. Theorem 3 follows easily from Theorem 2 in view of the elementary estimate

$$\|\zeta_n - \xi_n\|_\infty \le \max_{1\le k\le n} |X_k|.$$

Indeed, combining Lemma 3 in [13] with Lemma 4 in [10] it is clear that $U_n^{-1}\max_{1\le k\le n}|X_k|$ goes to zero in probability. So we only have to prove Theorem 2.

We shall prove that

$$b_n^{-1}\Big[\big(\xi_n, U_n\big) - \big(A\xi_n^{(\epsilon)}, |A|U_n(\epsilon)\big)\Big] \xrightarrow[n\to\infty]{\mathrm{Pr}} 0 \tag{19}$$

as $n \to \infty$, in the product space $C[0,1] \times \mathbb{R}$. If (19) is established, then Theorem 2 follows easily. Indeed, (19) yields that the asymptotic behavior of $\{U_n^{-1}\xi_n, n \in \mathbb{N}\}$ is the same as that of $\{(A/|A|)U_n^{-1}(\epsilon)\xi_n^{(\epsilon)}, n \in \mathbb{N}\}$. By Lemma 6 and Lemma 7:

$$\frac{A\xi_n^{(\epsilon)}}{|A|U_n(\epsilon)} \xrightarrow[n\to\infty]{\mathcal{D}} \frac{A}{|A|}W, \quad \text{in the space } C[0,1].$$

But $(A/|A|)W$ has the same distribution as W. So the proof reduces to (19).

Evidently, (19) follows from

$$b_n^{-1}\max_{t\in[0,1]}|\xi_n(t) - A\xi_n^{(\epsilon)}(t)| \xrightarrow[n\to\infty]{\mathrm{Pr}} 0 \tag{20}$$

and

$$b_n^{-1}\,|U_n - |A|U_n(\epsilon)| \xrightarrow[n\to\infty]{\mathrm{Pr}} 0. \tag{21}$$

First we prove (20). To this aim, introducing the elementary polygonal lines

$$e_{n,k}(t) = (nt - (k-1))\mathbf{1}_{[(k-1)/n,k/n]}(t) + \mathbf{1}_{(k/n,+\infty)}(t), \quad 1 \le k \le n,$$

we write for every $t \in [0,1]$, the expansion

$$\xi_n(t) = \sum_{k=1}^{n} X_k e_{n,k}(t) = \sum_{k=1}^{n} \sum_{i \in \mathbb{Z}} a_i \epsilon_{k-i} e_{n,k}(t)$$
$$= \sum_{i \in \mathbb{Z}} a_i \left(\sum_{k=1}^{n} \epsilon_{k-i} e_{n,k}(t) \right),$$

which leads naturally to introduce the partial sums processes

$$\xi_{n,i}^{(\epsilon)}(t) := \sum_{k=1}^{n} \epsilon_{k-i} e_{n,k}(t), \quad t \in [0,1],\ i \in \mathbb{Z},\ n \geq 1.$$

Thus

$$\xi_n = \sum_{i \in \mathbb{Z}} a_i \xi_{n,i}^{(\epsilon)}. \tag{22}$$

A priori, the series of functions (22) converges pointwise on $[0,1]$. In fact, this convergence holds also (almost surely) in the topology of $C[0,1]$, since by stationarity

$$\sum_{i \in \mathbb{Z}} |a_i| \, \mathbf{E} \left\| \xi_{n,i}^{(\epsilon)} \right\|_\infty = \mathbf{E} \left\| \xi_n^{(\epsilon)} \right\|_\infty \sum_{i \in \mathbb{Z}} |a_i| < \infty.$$

Denote $Z_{n,i} = b_n^{-1} \xi_{n,i}^{(\epsilon)}, i \in \mathbb{Z}, n \in \mathbb{N}$. To this collection of random elements in the Banach space $C[0,1]$ we shall apply Lemma 8. First we check Condition (i). We have by stationarity

$$\sup_{n \geq 1, i \in \mathbb{Z}} \mathbf{E} \, \|Z_{n,i}\|_\infty = \sup_{n \geq 1} \mathbf{E} \, \|Z_{n,0}\|_\infty = \sup_{n \geq 1} \left\| b_n^{-1} \xi_n^{(\epsilon)} \right\|_\infty.$$

To check the finiteness of this last supremum, we note first the stochastic boundedness of $(b_n^{-1} \xi_n^{(\epsilon)})_{n \geq 1}$ in $C[0,1]$ (combine (13) with Theorem 6). Then we apply Proposition 2 in [17] with $b_n = n^{1/2} L(n)$, L slowly varying.

Next we check (ii) of Lemma 8. We have

$$\|Z_{n,i} - Z_{n,j}\|_\infty = b_n^{-1} \left\| \sum_{k=1}^{n} (\epsilon_{k-i} - \epsilon_{k-j}) e_{n,k} \right\|_\infty = b_n^{-1} \max_{1 \leq k \leq n} |T_{i,j,k}|,$$

where

$$T_{i,j,k} = \sum_{\ell=1}^{k} (\epsilon_{\ell-i} - \epsilon_{\ell-j}).$$

Looking at the different possible configurations, we observe that

$$|T_{i,j,k}| \leq 2 \, |j-i| \max_{1-i \vee j \leq \ell \leq n-i \wedge j} |\epsilon_\ell|.$$

As i and j are fixed, we may assume that $n \geq \max(|i|, |j|)$, in which case we obtain

$$\max_{1 \leq k \leq n} |T_{i,j,k}| \leq 2 \, |j-i| \max_{-n < \ell \leq 2n} |\epsilon_\ell|.$$

By (11) $b_n^{-1}\max_{1\le \ell\le n}|\epsilon_\ell|$ goes to zero in probability and by stationarity, the same holds true for $b_n^{-1}\max_{-n<\ell\le 2n}|\epsilon_\ell|$, so Condition (ii) of Lemma 8 is satisfied. Hence, by this lemma we have

$$\left\|\sum_{i\in\mathbb{Z}} a_i Z_{n,i} - AZ_{n,0}\right\|_\infty \xrightarrow[n\to\infty]{\mathrm{Pr}} 0,$$

which coincides with (20).

Next we prove (21). We have

$$U_n^2 = \sum_{j=1}^N B_{m,j}^2 = \sum_{j=1}^N \left(\sum_{i\in\mathbb{Z}} a_i V_{n,j,i}\right)^2, \tag{23}$$

where $V_{n,j,i} = \sum_{\ell=(j-1)m+1}^{jm} \epsilon_{\ell-i}$ (recalling that $m = m(n)$). We shall check the conditions of Lemma 9 with $Z_{n,i}^{(j)} = b_n^{-1}V_{n,j,i}$, $n, j\in\mathbb{N}$, $i\in\mathbb{Z}$. For (i) we have to prove

$$b_n^{-1}\sup_{n,i}\mathbf{E}\left(\sum_{j=1}^N V_{n,j,i}^2\right)^{1/2} < \infty.$$

By stationarity this will follow from

$$b_n^{-1}\sup_{n\ge 1}\mathbf{E}\left(\sum_{j=1}^N V_{n,j,0}^2\right)^{1/2} < \infty. \tag{24}$$

Set for each $i\in\mathbb{Z}$,

$$\epsilon_i' = \epsilon_i \mathbf{1}_{\{|e_i|\le b_n\}} - \mathbf{E}\,\epsilon_i\mathbf{1}_{\{|\epsilon_i|\le b_n\}},\quad \epsilon_i'' = \epsilon_i\mathbf{1}_{\{|\epsilon_i|>b_n\}} - \mathbf{E}\,\epsilon_i\mathbf{1}_{\{|\epsilon_i|>b_n\}}$$

and define $V_{n,j,i}'$, $V_{n,j,i}''$ by substituting respectively ϵ by ϵ', ϵ'' in the definition of $V_{n,j,i}$. As $\mathbf{E}\,\epsilon_i = 0$, $\epsilon_i = \epsilon_i' + \epsilon_i''$ and we have

$$\mathbf{E}\left(\sum_{j=1}^N V_{n,j,0}^2\right)^{1/2} \le T_n' + T_n'',$$

where

$$T_n' = \mathbf{E}\left(\sum_{j=1}^N V_{n,j,0}'^2\right)^{1/2},\quad T_n'' = \mathbf{E}\left(\sum_{j=1}^N V_{n,j,0}''^2\right)^{1/2}.$$

Using Jensen inequality and recalling that $Nm\le n$ we get

$$b_n^{-1}T_n' \le b_n^{-1}\left(\sum_{j=1}^N \mathbf{E}\,V_{n,j,0}'^2\right)^{1/2} \le b_n^{-1}(n\mathbf{E}\,\epsilon_1'^2)^{1/2}.$$

Hence $\sup_n b_n^{-1} T'_n < \infty$ by (12). Next, by comparison of the norms $\ell^1(N)$ and $\ell^2(N)$, we obtain

$$T''_n \leq \sum_{j=1}^{N} \mathbf{E}\left|V''_{n,j,0}\right| \leq n\mathbf{E}\,|\epsilon''_1|.$$

As $\mathbf{E}\,|\epsilon''_1| \leq 2\mathbf{E}\,|\epsilon_1|\,\mathbf{1}\{|\epsilon_1| > b_n\}$, $b_n^{-1}T''_n$ converges to zero by Lemma 4 which completes the verification of (24) and of Condition (i).

Next we check Condition (ii) of Lemma 4, that is we have to establish the convergence

$$b_n^{-2}\sum_{j=1}^{N}(V_{n,j,i} - V_{n,j,\ell})^2 \xrightarrow[n\to\infty]{\Pr} 0.$$

By stationarity and a simple chaining argument it is enough to prove that

$$b_n^{-2}\sum_{j=1}^{N}(V_{n,j,1} - V_{n,j,0})^2 \xrightarrow[n\to\infty]{\Pr} 0.$$

This follows from

$$b_n^{-2}\sum_{j=1}^{N}\epsilon_{jm}^2 \xrightarrow[n\to\infty]{\Pr} 0.$$

As the random vectors $(\epsilon_i)_{1\leq i\leq N}$ and $(e_{jm})_{1\leq j\leq N}$ have the same distribution, it is enough to check that

$$b_n^{-2}\sum_{i=1}^{N}\epsilon_i^2 \xrightarrow[n\to\infty]{\Pr} 0. \tag{25}$$

Accounting (13), this reduces to

$$\frac{V_N^2(\epsilon)}{V_n^2(\epsilon)} \xrightarrow[n\to\infty]{\Pr} 0.$$

Noting that the non negative random variables $Y_{n,i} = V_n^{-2}(\epsilon)\epsilon_i^2$, $i = 1,\ldots,n$ have identical distribution and that $\sum_{i=1}^n Y_{n,i} = 1$, it follows that $\mathbf{E}\,Y_{n,i} = 1/n$ and consequently

$$\mathbf{E}\,\frac{V_N^2(\epsilon)}{V_n^2(\epsilon)} = \frac{N}{n}.$$

This implies the convergence (25) since N/n tends to zero. By Lemma 9 we conclude (21). This completes the proof of Theorem 2.

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