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Stochastic domination for the last passage percolation model

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Abstract

A competition model on $\mathbb{Z}_+^2$ governed by directed last passage percolation is considered. A stochastic domination argument between subtrees of the last passage percolation tree is put forward.

Résumé

Le modèle de percolation de dernier passage dans le quadrant $\mathbb{Z}_+^2$ produit un arbre aléatoire, appelé arbre de percolation de dernier passage. Certains de ses sous-arbres peuvent être comparés stochastiquement.

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1 Introduction

The directed last passage percolation model goes back to the original work of Rost [8] in the case of i.i.d. exponential weights. In this paper, Rost proved a shape theorem for the infected region and exhibited for the first time a link with the one-dimensional totally asymmetric simple exclusion process (TASEP). A background on exlusion processes can be found in the book [5] of Liggett. Since then, this link has been done into details by Ferrari and its coauthors [1, 2, 3] to obtain aymptotic directions and related results for competition interfaces. Other results have been obtained in the case of i.i.d. geometric weights : see Johansson [4]. For i.i.d. weights but with general weight distribution, Martin [6] proved a shape theorem and described the behavior of the shape function close to the boundary. See also the survey [7].

Let us consider $\Omega = [0,\infty)^{\mathbb{Z}^2}$ refered as the configuration space and endowed with a Borel probability measure $\mathbb{P}$. All throughout this paper, $\mathbb{P}$ is assumed translation-invariant : for all $a \in \mathbb{Z}^2$,

$$\mathbb{P} = \mathbb{P} \circ \tau_a^{-1} \ ,$$

where τ_a denotes the translation operator on Ω defined by $\tau_a(\omega) = \omega(a+\cdot)$. This is the only assumption about the probability measure $\mathbb{P}$. We are interested in the behavior of optimal paths from the origin to a site $z \in \mathbb{Z}_+^2$. The collection of optimal paths forms the last passage percolation tree $\mathcal{T}$. In this paper, a special attention is paid to the subtree of $\mathcal{T}$ rooted at $(1,1)$: see Figure 1.

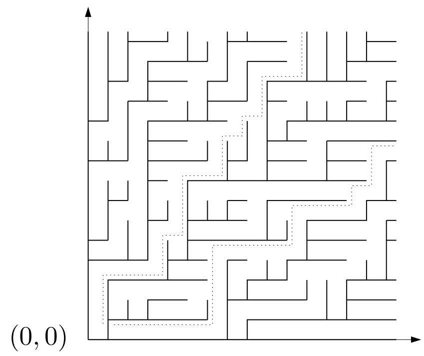

Figure 1: An example of the last passage percolation tree on the set $[0;15]^2$. The subtree rooted at $(1,1)$ is surrounded by dotted lines. Here, the upper dotted line corresponds to the competition interface studied by Ferrari and Pimentel in [3].

Our goal is to stochastically dominate subtrees of the last passage percolation tree by the one rooted at $(1,1)$. Our results (Theorems 2 and 3) essentially rely on elementary properties of the last passage percolation model; its directed nature and the positivity of weights.

The paper is organized as follows. In the rest of this section, optimal paths and the last passage percolation tree $\mathcal{T}$ are precisely defined. The growth property which allows us to compare subtrees of $\mathcal{T}$ is introduced. Theorems 2 and 3 are stated and commented in Section 2. They are proved in Section 3.

1.1 Paths, low-optimality, percolation tree

We will focus on (up-right oriented only) *paths* which can be defined as sequences (finite or not) $\gamma = (z_0, z_1, \ldots)$ of sites $z_i \in \mathbb{Z}^2$ such that $z_{i+1} - z_i = (1,0)$ or $(0,1)$.

For a given configuration ω, we define the *length* of a path γ as

$$\omega(\gamma) = \sum_{z \in \gamma} \omega(z).$$

If Γ_z is the (finite) set of paths from $(0,0)$ to z, a path $\gamma \in \Gamma_z$ is *ω-optimal* if its length $\omega(\gamma)$ is maximal on Γ_z. The quantity $\max_{\gamma \in \Gamma_z} \omega(\gamma)$ is known as the *last passage time* at z. To avoid questions on uniqueness of optimal paths, it is convenient to call *low-optimal* the optimal path below all the others.

Proposition 1. *Given $\omega \in \Omega$, each Γ_z contains a (unique) low-optimal path denoted by γ_z^ω.*

Proof We can assume that $\mathrm{Card}(\Gamma_z) \geq 2$. Given $\omega \in \Omega$, consider two arbitrary optimal paths γ, γ' of Γ_z. If they have no common point (except endpoints $(0,0)$ and z), then one path is below the other. If γ and γ' meet in sites, say $u_1, \ldots, u_k$, it's easy to see that the path which consists in concatenation of lowest subpaths of γ, γ' between consecutive u_i, u_{i+1} is also an optimal path of Γ_z. This procedure can be (finitely) repeated to reach the low-optimal path of Γ_z for the configuration ω. ■

In literature, optimal paths are generally unique and called *geodesics*. This is the case when $\mathbb{P}$ is a product measure over $\mathbb{Z}^2$ of non-atomic laws. Here, low-optimality ensures uniqueness without particular restriction and Proposition 1 allows then to define the (last passage) percolation tree $\mathcal{T}^\omega$ as the collection of low-optimal paths γ_z^ω for all $z \in \mathbb{Z}_+^2$. Moreover, the subtree of $\mathcal{T}^\omega$ rooted at z is denoted by $\mathcal{T}_z^\omega$.

1.2 Growth property

Let us introduce the set $\mathbb{T}$ of all substrees of $\mathcal{T}$:

$$\mathbb{T} = \{\mathcal{T}_z^\omega \,:\, z \in \mathbb{Z}_+^2,\, \omega \in \Omega\}.$$

For a tree $T \in \mathbb{T}$, $r(T)$ and $V(T)$ denote respectively its root and its vertex set.

Definition 1. *A subset A of $\mathbb{T}$ satisfies the* growth property *if*

$$(T \in A,\ T' \in \mathbb{T},\ V(T) - r(T) \subset V(T') - r(T')) \implies T' \in A. \quad (1)$$

For example, if $k \in \mathbb{Z}_+ \cup \{\infty\}$, the set $\{T \in \mathbb{T} : \text{Card}V(T) \geq k\}$ satifies the growth property. But so does not the set

$$\{T \in \mathbb{T} : T \text{ have at least two infinite branches}\} .$$

Indeed, the partial ordering on the set $\mathbb{T}$ induced by Definition 1 does not take into account the graph structure of trees.

2 Stochastic domination

The following results compare subtrees of the last passage percolation tree through subsets of $\mathbb{T}$ satisfying the growth property.

Theorem 2. *Let $a \in \mathbb{Z}_+^2$ and a subset A of $\mathbb{T}$ satisfying the growth property* (1)*. Set also $\Omega^a = \left\{\omega \in \Omega : a \text{ belongs to } \gamma^\omega_{a+(1,0)} \text{ and } \gamma^\omega_{a+(0,1)}\right\}$. Then,*

$$\mathbb{P}\left(\mathcal{T}_{a+(1,1)} \in A, \Omega^a\right) \leq \mathbb{P}\left(\mathcal{T}_{(1,1)} \in A, \tau_a(\Omega^a)\right) .$$

In particular, if $\mathbb{P}$ is in addition a product measure, we have

$$\mathbb{P}\left(\mathcal{T}_{a+(1,1)} \in A \mid \Omega^a\right) \leq \mathbb{P}\left(\mathcal{T}_{(1,1)} \in A\right) ,$$

To illustrate the meaning of this result, assume that the vertices of $\mathcal{T}_{(1,1)}$ are painted in blue and those of $\mathcal{T}_{(2,0)}$ and $\mathcal{T}_{(0,2)}$ in red. This random coloration leads to a competition of colors. The red area is necessarily unbounded since the model forces every vertice $(x, 0)$ or $(0, x)$ with $x \in \{2, 3, \ldots\}$ to be red. But the blue area can be bounded. Now, consider $a \in \mathbb{Z}_+^2$ and the same way to color but only in the quadrant $a + \mathbb{Z}_+^2$: this time, the blue area consists of the vertices of $\mathcal{T}_{a+(1,1)}$ and the red one of

$$\mathcal{T}_{a+(x,0)} \text{ and } \mathcal{T}_{a+(0,x)}, \text{ for } x \geq 2 .$$

Roughly speaking, Theorem 2 says that, conditionally to Ω^a, the competition is harder for the latter blue area.

The proof of Theorem 2 can be summed up as follows. From a configuration ω, a new one which is a perturbed translation of ω, namely $\omega^a = \tau_a(\omega) + \varepsilon$ is built in order to satisfy

$$\mathcal{T}^\omega_{a+(1,1)} = a + \mathcal{T}^{\omega^a}_{(1,1)}.$$

But ε is chosen such that for $\omega \in \Omega^a$, we have $V(\mathcal{T}^{\omega^a}_{(1,1)}) \subset V(\mathcal{T}^{\tau_a(\omega)}_{(1,1)})$ and the growth property leads to

$$\mathcal{T}^\omega_{a+(1,1)} \in A \implies \mathcal{T}^{\tau_a(\omega)}_{(1,1)} \in A.$$

It remains then to use the translation invariance of $\mathbb{P}$ to get the result.

The next result suggests a second stochastic domination argument in the spirit of Theorem 2.

Theorem 3. *Let $m \in \mathbb{N}$ and a subset A satisfying the growth property* (1)*. Set $\Omega_m = \{\omega : \gamma^{\omega}_{(m,1)} = ((0,0),(1,0),\ldots,(m,0),(m,1))\}$. Then*

$$\mathbb{P}(\mathcal{T}_{(m,1)} \in A, \Omega_m) \leq \mathbb{P}(\mathcal{T}_{(1,1)} \in A, \Omega_1) \ . \tag{2}$$

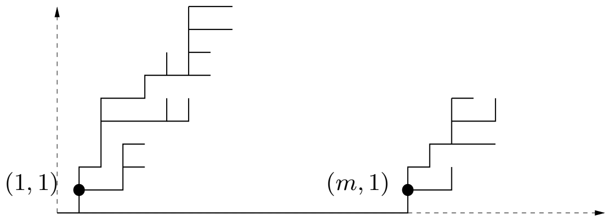

Figure 2: Are represented the subtrees of the last passage percolation tree rooted at sites $(1,1)$ and $(m,1)$, for a configuration $\omega \in \Omega^1 \cap \Omega^m$.

Now some comments are needed.

- Note that $\Omega_1 = \{\omega : \omega(1,0) > \omega(0,1)\}$.
- It is worth pointing out here that Theorem 3 is, up to a certain extend, better than Theorem 2. If $a = (m,0)$ then the events Ω^a and Ω_m are equal and the probablity $\mathbb{P}(\mathcal{T}_{a+(1,1)} \in A, \Omega^a)$ can be splitted into

$$\mathbb{P}\left(\mathcal{T}_{(m+1,1)} \in A,\ \Omega_m,\ \omega(m+1,0) < \omega(m,1)\right) \tag{3}$$

  and

$$\mathbb{P}\left(\mathcal{T}_{(m+1,1)} \in A,\ \Omega_m,\ \omega(m+1,0) \geq \omega(m,1)\right) \ . \tag{4}$$

  On the event $\{\omega(m+1,0) < \omega(m,1)\}$, $\mathcal{T}_{(m+1,1)}$ is as a subtree of $\mathcal{T}_{(m,1)}$. Hence, if A satisfies the growth property (1) then $\mathcal{T}_{(m+1,1)} \in A$ forces $\mathcal{T}_{(m,1)} \in A$. It follows that (3) is bounded by $\mathbb{P}(\mathcal{T}_{(m,1)} \in A, \Omega_m)$ which is at most $\mathbb{P}(\mathcal{T}_{(1,1)} \in A, \Omega_1)$ by Theorem 3.
  On the other hand, $\{\Omega_m, \omega(m+1,0) \geq \omega(m,1)\}$ is included in Ω_{m+1}. Consequently, (4) is bounded by $\mathbb{P}(\mathcal{T}_{(m+1,1)} \in A, \Omega_{m+1})$, and also by $\mathbb{P}(\mathcal{T}_{(1,1)} \in A, \Omega_1)$ by Theorem 3 again.
  Combining these bounds, we get

$$\mathbb{P}(\mathcal{T}_{a+(1,1)} \in A,\ \Omega^a) \leq 2\,\mathbb{P}(\mathcal{T}_{(1,1)} \in A,\ \Omega_1) \ .$$

  To sum up, whenever $2\,\mathbb{P}(\mathcal{T}_{(1,1)} \in A, \Omega_1)$ is smaller than $\mathbb{P}(\mathcal{T}_{(1,1)} \in A)$ (this is the case when $\mathbb{P}$ and A are invariant by the symmetry with respect to the diagonal $x = y$), Theorem 2 with $a = (m,0)$ can be obtained as a consequence of Theorem 3.
- Let us remark that further work seems to lead to the following improvement of Theorem 3: the application

$$m \ \mapsto \mathbb{P}\left(\mathcal{T}_{(m,1)} \in A\ ,\ \Omega^m\right)$$

  should be non increasing.

– Finally, by symmetry, Theorem 3 obviously admits an analogous version on the other axis. Roughly speaking, the subtree of the last passage percolation tree rooted at the site $(1, m)$ is stochastically dominated by the one rooted at $(1, 1)$.

Here are two situations in which Theorem 3 can be used.
An infinite low-optimal path is said non trivial if it does not coincide with one of the two axes $\mathbb{Z}_+(1,0)$ and $\mathbb{Z}_+(0,1)$. If the set $V(\mathcal{T}_{(1,1)})$ is unbounded (which can be referred as "coexistence") then, since each vertex in a subtree has a bounded number of children (in fact, at most 2), the tree $\mathcal{T}_{(1,1)}$ contains an infinite low-optimal path. So, if we set

$$Coex = \{\mathrm{Card} V(\mathcal{T}_{(1,1)}) = \infty\},$$

then

$$\mathbb{P}\,(Coex) > 0 \implies \mathbb{P}\left(\begin{array}{c}\text{there exists a non trivial,}\\ \text{infinite low-optimal path}\end{array}\right) > 0\ . \tag{5}$$

Conversely, assume that $\mathbb{P}(Coex)$ is zero. Since the set

$$\{T \in \mathbb{T}\ :\ \mathrm{Card}(V(T)) = \infty\}$$

satisfies the growth property, Theorem 3 implies that for all $m \in \mathbb{N}$

$$\mathbb{P}\,\big(\mathrm{Card}(V(\mathcal{T}_{(m,1)})) = \infty,\ \Omega_m\big) = 0\ .$$

Hence, $\mathbb{P}$ –a.s., each subtree coming from the axis $\mathbb{Z}_+(1,0)$ is finite. This result can be generalized to the two axes $\mathbb{Z}_+(1,0)$ and $\mathbb{Z}_+(0,1)$ by symmetry. Then, $\mathbb{P}$ –a.s., there is no non trivial, infinite low-optimal path and (5) becomes an equivalence.

Now, Set

$$\Delta_n = \{(x,y) \in \mathbb{Z}_+^2\ :\ x + y = n\},$$

and let us denote by α_n the (random) number of vertices of $\mathcal{T}_{(1,1)}$ meeting Δ_n:

$$\alpha_n = \mathrm{Card}\,\big(V(\mathcal{T}_{(1,1)}) \cap \Delta_n\big)\ .$$

The event $Coex$ can be written $\bigcap_{n\in\mathbb{N}}\{\alpha_n > 0\}$. We say there is "strong coexistence" if

$$\limsup_{n\to\infty} \frac{\alpha_n^\omega}{n} > 0\ .$$

In a future work, Theorem 3 is used so as to give sufficient conditions ensuring strong coexistence with positive probability.

3 Proofs

3.1 Proof of Theorem 2

Recall that γ_z^ω denotes the low-optimal path from 0 to z for ω.

I/ Let ω and $\omega+\varepsilon$ be configurations where ε is a vanishing configuration except on the axes $\mathbb{Z}_+(1,0)$ and $\mathbb{Z}_+(0,1)$ i.e $\varepsilon(x,y)=0$ if $xy\neq 0$. We shall show that if ε satifies

$$\varepsilon(0,1)+\varepsilon(0,2)\geq\varepsilon(1,0) \text{ and } \varepsilon(1,0)+\varepsilon(2,0)\geq\varepsilon(0,1), \tag{6}$$

then

$$V(\mathcal{T}^{\omega+\varepsilon}_{(1,1)})\subset V(\mathcal{T}^{\omega}_{(1,1)}). \tag{7}$$

Let $z\in V(\mathcal{T}^{\omega+\varepsilon}_{(1,1)})$. By definition, z is a vertex such that the low-optimal path $\gamma_z^{\omega+\varepsilon}$ contains $(1,1)$; we have to show that $z\in V(\mathcal{T}^{\omega}_{(1,1)})$ i.e. the low-optimal path γ_z^{ω} also contains $(1,1)$.
There is nothing to prove if $\gamma_z^{\omega+\varepsilon}=\gamma_z^{\omega}$, so we assume that $\gamma_z^{\omega+\varepsilon}\neq\gamma_z^{\omega}$. By additivity of $\omega\mapsto\omega(\gamma)$ and optimality of $\gamma_z^{\omega+\varepsilon}$ and γ_z^{ω}, we have

$$\begin{aligned}\varepsilon(\gamma_z^{\omega}) &= (\omega+\varepsilon)(\gamma_z^{\omega})-\omega(\gamma_z^{\omega})\\ &\leq (\omega+\varepsilon)(\gamma_z^{\omega+\varepsilon})-\omega(\gamma_z^{\omega})\\ &= \omega(\gamma_z^{\omega+\varepsilon})+\varepsilon(\gamma_z^{\omega+\varepsilon})-\omega(\gamma_z^{\omega})\\ &\leq \varepsilon(\gamma_z^{\omega+\varepsilon}).\end{aligned} \tag{8}$$

Note also that by low-optimality of $\gamma_z^{\omega+\varepsilon}$ and γ_z^{ω}, we have

$$\big((\omega+\varepsilon)(\gamma_z^{\omega+\varepsilon})=(\omega+\varepsilon)(\gamma_z^{\omega}) \text{ and } \omega(\gamma_z^{\omega+\varepsilon})=\omega(\gamma_z^{\omega})\big)\Longrightarrow\gamma_z^{\omega+\varepsilon}=\gamma_z^{\omega}.$$

Since $\gamma_z^{\omega+\varepsilon}$ and γ_z^{ω} are different, this allows us to strengthen (8) :

$$\varepsilon(\gamma_z^{\omega})<\varepsilon(\gamma_z^{\omega+\varepsilon}). \tag{9}$$

Now, it's got to be one thing or the other:

– Either the site $(1,0)$ belongs to $\gamma_z^{\omega+\varepsilon}$. In this case, the right hand side of (9) which becomes $\varepsilon(0,0)+\varepsilon(1,0)$ and the strict inequality imply that γ_z^{ω} can not run through $(1,0)$. Moreover, if γ_z^{ω} ran through $(0,2)$, we would have

$$\varepsilon(0,0)+\varepsilon(0,1)+\varepsilon(0,2)\leq\varepsilon(\gamma_z^{\omega})<\varepsilon(0,0)+\varepsilon(1,0),$$

but this would be in contradiction with (6). We conclude that γ_z^{ω} must run through $(0,1)$ and $(1,1)$.

– Or the site $(0,1)$ belongs to $\gamma_z^{\omega+\varepsilon}$, and symmetrically we conclude that, if (6) hold, γ_z^{ω} runs through $(1,0)$ and $(1,1)$.

To sum up, if ε satisfies (6) then (7) holds. The conditions (6) can be understood as follows; the second one (for example) prevents the set $V(\mathcal{T}_{(2,0)})$ to drop vertices in favour of $V(\mathcal{T}_{(1,1)})$ passing from ω to $\omega+\varepsilon$.

II/ For given configuration $\omega\in\Omega$ and site $a\in\mathbb{Z}_+^2$, we construct a new configuration ω^a such that

$$\forall z\in\mathbb{Z}_+^2,\quad \omega^a\left(\gamma_z^{\omega^a}\right)=\omega\left(\gamma_{a+z}^{\omega}\right). \tag{10}$$

The idea of the constuction is to translate ω from a to the origin and to modify then weights on the axes : more precisely, set

$$\omega^a(z) = \begin{cases} \omega(\gamma_a^\omega) & \text{if } z=(0,0); \\ \omega(\gamma_{a+z}^\omega) - \omega(\gamma_{a+z-(1,0)}^\omega) & \text{if } z=(x,0) \text{ with } x\in\mathbb{N}; \\ \omega(\gamma_{a+z}^\omega) - \omega(\gamma_{a+z-(0,1)}^\omega) & \text{if } z=(0,y) \text{ with } y\in\mathbb{N}; \\ \omega(a+z) & \text{otherwise.} \end{cases}$$

Let $\bar z$ be the latest site of $a+(\mathbb{Z}_+(1,0)\cup\mathbb{Z}_+(0,1))$ whereby γ_{a+z}^ω passes. The configuration ω^a is defined so as to the last passage time to $\bar z$ for ω is equal to the last passage time to $\bar z - a$ for ω^a, i.e. $\omega(\gamma_{\bar z}^\omega) = \omega^a(\gamma_{\bar z - a}^{\omega^a})$. Combining with $\omega^a(\cdot) = \omega(a+\cdot)$ on $a+\mathbb{N}^2$, the identity (10) follows. Finally, by low-optimality, the translated path $a+\gamma_z^{\omega^a}$ coincides with the restriction of γ_{a+z}^ω to the quadrant $a+\mathbb{N}^2$. See Figure 3.

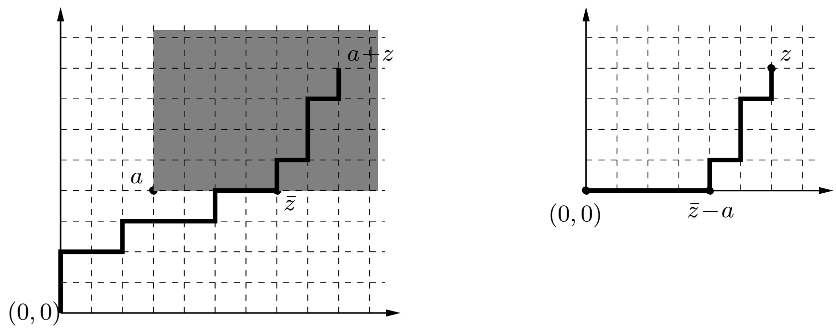

Figure 3: To the left, the low-optimal path to $a+z$ for a given configuration ω is represented. Let us denote by $\bar z$ the latest site of $a+(\mathbb{Z}_+(1,0)\cup\mathbb{Z}_+(0,1))$ whereby γ_{a+z}^ω passes. To the right, the low-optimal path to z for the corresponding configuration ω^a is represented.

In particular, we can write with some abuse of notation

$$a + \mathcal{T}_{(1,1)}^{\omega^a} = \mathcal{T}_{a+(1,1)}^{\omega}. \tag{11}$$

The induction formula's

$$\omega(\gamma_u^\omega) = \max(\omega(\gamma_{u-(1,0)}^\omega), \omega(\gamma_{u-(0,1)}^\omega)) + \omega(u), \tag{12}$$

allows to rewrite the configuration ω^a:

$$\omega^a = \tau_a(\omega) + \varepsilon, \tag{13}$$

where ε is defined on the axes by

$$\begin{array}{rcll} \varepsilon(0,0) & = & \max\Big(\omega(\gamma^{\omega}_{a-(1,0)}),\omega(\gamma^{\omega}_{a-(0,1)})\Big) & \\ \varepsilon(x,0) & = & \max\Big(0,\omega(\gamma^{\omega}_{a+(x,-1)})-\omega(\gamma^{\omega}_{a+(x-1,0)})\Big) & (x\in\mathbb{N}) \quad (14)\\ \varepsilon(0,y) & = & \max\Big(0,\omega(\gamma^{\omega}_{a+(-1,y)})-\omega(\gamma^{\omega}_{a+(0,y-1)})\Big) & (y\in\mathbb{N}) \quad (15)\\ \varepsilon(x,y) & = & 0 \quad \text{otherwise}\ . & \end{array}$$

III/ Consider a and Ω^a as in the statement of Theorem 2. Let $\omega \in \Omega^a$ so that the length $\omega(\gamma^{\omega}_a)$ is bigger than $\omega(\gamma^{\omega}_{a+(1,-1)})$ and $\omega(\gamma^{\omega}_{a+(-1,1)})$. It follows from (14) and (15) that $\varepsilon(1,0)=\varepsilon(0,1)=0$. Conditions (6) are then trivially satisfied so that (7) holds for ω and also for $\tau_a(\omega)$:

$$V(\mathcal{T}^{\tau_a(\omega)+\varepsilon}_{(1,1)}) \subset V(\mathcal{T}^{\tau_a(\omega)}_{(1,1)})\ .$$

Combined with (11) and (13), this leads to

$$V(\mathcal{T}^{\omega}_{a+(1,1)}) - a \subset V(\mathcal{T}^{\tau_a(\omega)}_{(1,1)}).$$

Now, if A satisfies the growth property (1) then

$$\mathcal{T}^{\omega}_{a+(1,1)} \in A \implies \mathcal{T}^{\tau_a(\omega)}_{(1,1)} \in A.$$

To summarize, we have $\{\mathcal{T}_{a+(1,1)} \in A\} \subset \tau_a^{-1}\{\mathcal{T}_{(1,1)} \in A\}$ on Ω^a, and since $\mathbb{P}$ is translation-invariant and $\Omega^a = \tau_a^{-1}(\tau_a(\Omega^a))$, we conclude that

$$\begin{array}{rcl} \mathbb{P}(\mathcal{T}_{a+(1,1)} \in A, \Omega^a) & \leq & \mathbb{P}\left(\tau_a^{-1}\{\mathcal{T}_{(1,1)} \in A\}, \Omega^a\right) \\ & = & \mathbb{P}(\mathcal{T}_{(1,1)} \in A, \tau_a(\Omega^a)). \end{array}$$

The first part of Theorem 2 is proved. In order to prove the second part, let us assume $\mathbb{P}$ is a product measure. It suffices to remark the events $\tau_a(\Omega^a)$ which means both low-optimal paths from $-a$ to $(1,0)$ and $(0,1)$ run through the origin, and $\{\mathcal{T}_{(1,1)} \in A\}$ are independent. Actually, the random variable $\omega(0,0)$ is the only weight of $\mathbb{Z}^2_+$ of which $\tau_a(\Omega^a)$ depends on, and it is involved in all optimal paths coming from the origin. So, it does not affect the event $\{\mathcal{T}_{(1,1)} \in A\}$.

3.2 Proof of Theorem 3

I/ Let $\omega \in \Omega_1$ and $\omega+\varepsilon$ be two configurations where ε is a vanishing configuration except on the axis $\mathbb{Z}_+(0,1)$: $\varepsilon(x,y)=0$ whenever $x>0$. We also assume that ω and ε verify $\omega(1,0) > \omega(0,1)+\varepsilon(0,1)$ (i.e. $\omega+\varepsilon \in \Omega_1$). The goal of the first step consists in stating:

$$V(\mathcal{T}^{\omega+\varepsilon}_{(1,1)}) \subset V(\mathcal{T}^{\omega}_{(1,1)})\ . \quad (16)$$

Let z be a vertex such that the low-optimal path $\gamma_z^{\omega+\varepsilon}$ contains $(1,1)$. If the low-optimal paths γ_z^{ω} and $\gamma_z^{\omega+\varepsilon}$ are different then it follows as for (9)):

$$\varepsilon(\gamma_z^{\omega}) < \varepsilon(\gamma_z^{\omega+\varepsilon}) \ .$$

Henceforth, the condition $\omega+\varepsilon \in \Omega_1$ implies that $\gamma_z^{\omega+\varepsilon}$ runs through $(1,0)$ and leads to a contradiction:

$$\varepsilon(0,0) \leq \varepsilon(\gamma_z^{\omega}) < \varepsilon(\gamma_z^{\omega+\varepsilon}) = \varepsilon(0,0) \ .$$

So, γ_z^{ω} and $\gamma_z^{\omega+\varepsilon}$ are equal, which implies z is a vertex of $\mathcal{T}_{(1,1)}^{\omega}$. Relation (16) is proved. It is worth to note that condition $\omega+\varepsilon \in \Omega_1$ ensures that the random interface between sets $V(\mathcal{T}_{(1,1)})$ and $V(\mathcal{T}_{(2,0)})$ remains unchanged if we add ε to ω. Hence, the set $V(\mathcal{T}_{(1,1)})$ can only decrease.

II/ Let ω be a configuration and $b=(m-1,0)$. In the spirit of the proof of Theorem 2, a configuration ω^b is built by translating ω by vector $-b$ and preserving the last passage percolation tree structure. The right construction is the following:

$$\omega^b(z) = \begin{cases} \omega(\gamma_b^{\omega}) & \text{if } z=(0,0); \\ \omega(\gamma_{b+z}^{\omega}) - \omega(\gamma_{b+z-(0,1)}^{\omega}) & \text{if } z \in \{0\}\times\mathbb{N}; \\ \omega(b+z) & \text{otherwise.} \end{cases}$$

By construction, the configuration ω^b satisfies $\omega^b(\gamma_z^{\omega^b}) = \omega(\gamma_{b+z}^{\omega})$, for all $z \in \mathbb{Z}_+^2$. Thus, we can deduce from low-optimality:

$$b + \mathcal{T}_{(1,1)}^{\omega^b} = \mathcal{T}_{b+(1,1)}^{\omega} \ . \tag{17}$$

The induction formula's (12) allows to write for all $z \in \mathbb{Z}_+^2$,

$$\omega^b(z) = \tau_b(\omega)(z) + \varepsilon(z) \ ,$$

with

$$\varepsilon(z) = \begin{cases} \omega(\gamma_{b-(1,0)}^{\omega}) & \text{if } z=(0,0); \\ \max\Big(\omega(\gamma_{b+z-(1,0)}^{\omega}) - \omega(\gamma_{b+z-(0,1)}^{\omega}), 0\Big) & \text{if } z \in \{0\}\times\mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

Besides,

$$\begin{aligned} \omega \in \Omega_m \quad & \iff \quad \omega(\gamma_{b+(0,1)}^{\omega}) < \omega(\gamma_{b+(1,0)}^{\omega}) \\ & \iff \quad \omega^b(0,1) + \omega(\gamma_b^{\omega}) < \omega(\gamma_b^{\omega}) + \omega(b+(1,0)) \\ & \iff \quad \omega^b(0,1) < \omega^b(1,0) \\ & \iff \quad \omega^b \in \Omega_1 \ . \end{aligned} \tag{18}$$

III/ Given $\omega \in \Omega_m$, equivalence (18) implies $\omega^b = \tau_b(\omega) + \varepsilon \in \Omega_1$. As a by-product, we have $\tau_b(\omega) \in \Omega_1$ and from (16) and (17), we deduce

$$V(\mathcal{T}^{\omega}_{b+(1,1)}) - b \subset V(\mathcal{T}^{\omega^b}_{(1,1)}) \subset V(\mathcal{T}^{\tau_b(\omega)}_{(1,1)}) \ .$$

If $A \subset \mathbb{T}$ satisfies the growth property (1) then

$$\left(\omega \in \Omega_m \ \text{ and } \ \mathcal{T}^{\omega}_{(m,1)} \in A\right) \implies \left(\tau_b(\omega) \in \Omega_1 \ \text{ and } \ \mathcal{T}^{\tau_b(\omega)}_{(1,1)} \in A\right) \ .$$

Finally, (2) easily follows from the translation invariance of the probability measure $\mathbb{P}$.

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