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Periods of mixed Tate motives, examples, l-adic side

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Abstract

One hopes that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}$ is generated by values of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ in one forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\vec{01}$ to $\vec{10}$. These numbers are also called multi zeta values. In this note we give a sketch of a proof, assuming motivic formalism, that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ in one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\vec{01}$ to $\vec{10}$, which are unramified everywhere. We shall discuss also l-adic analog of this result and also some other examples.

Résumé

Résumé en français à compléter

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0 Introduction

0.1. One hopes that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}$ is generated by values of iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ in one forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. These numbers are also called multi zeta values. In this note we give a sketch of a proof, assuming motivic formalism, that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,-1,\infty\}$ in one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are unramified everywhere.

We give also a criterium when a linear combination with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,-1,\infty\}$ in one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ is unramified everywhere. Such result may be useful even if finally one shows that iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ in one forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ generate the $\mathbb{Q}$-algebra of mixed Tate motives over $Spec\,\mathbb{Z}$.

These results have their analogs in l-adic realizations. In fact we shall study l-adic situation first and in more details. The l-adic situation is easier conceptually, because the Galois group G_K of a number field K and its various weighted Tate $\mathbb{Q}_l$-completions replace the motivic fundamental group of the category of mixed Tate motives over $Spec\,\mathcal{O}_{K,S}$, which is perhaps still a conjectural object.

We shall consider weighted Tate representations of $\pi_1^t(Spec\,\mathcal{O}_{K,\mathcal{S}};Spec\bar{K})$ in finite dimensional $\mathbb{Q}_l$-vector spaces. The universal proalgebraic group over $\mathbb{Q}_l$ by which such representations factorize we shall denote by $\mathcal{G}(\mathcal{O}_{K,\mathcal{S}};l)$. The kernel of the projection $\mathcal{G}(\mathcal{O}_{K,\mathcal{S}};l)\to\mathbb{G}_m$ we denote by $\mathcal{U}(\mathcal{O}_{K,\mathcal{S}};l)$. The associated graded Lie algebra of $\mathcal{U}(\mathcal{O}_{K,\mathcal{S}};l)$ with respect of the weight filtration we denote by $gr_W Lie\mathcal{U}(\mathcal{O}_{K,\mathcal{S}};l)$.

We assume that $\mathcal{S}$ contains all finite places of K lying over (l). Then the group $\mathcal{G}(\mathcal{O}_{K,\mathcal{S}};l)$ is isomorphic to the conjectural motivic fundamental group of the tannakian category of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,\mathcal{S}}$ tensored with $\mathbb{Q}_l$ (see [5] and [6]). There it is also considered the case when $\mathcal{S}$ does not contain all finite places of K lying over (l). However the construction is decidely more complicated and we do not understand it very well. As we consider also the case of arbitrary $\mathcal{S}$ we shall modify slightly the graded Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,\mathcal{S}};l)$ to serve our proposal. This is described briefly below.

Let S be a finite set of finite places of K. A non trivial l-adic weighted Tate representation of G_K is ramified at all finite places of K which lie over (l). Therefore we must consider the weighted Tate $\mathbb{Q}_l$-completion of $\pi_1^t(Spec\,\mathcal{O}_{K,\mathcal{S}};Spec\bar{K})$, where $\mathcal{S}$ is a union of S and all finite places of K lying over (l). This has an effect that the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,\mathcal{S}};l)$ has more generators in degree 1 than the corresponding Lie algebra of the tannakian category of mixed Tate motives over $Spec\,\mathcal{O}_{K,S}$. To get rid of these additional generators in degree 1 we shall define a homogenous Lie ideal $\langle\mathfrak{l}\mid l\rangle_{K,S}$ of $gr_W Lie\mathcal{U}(\mathcal{O}_{K,\mathcal{S}};l)$ and then the quotient Lie algebra

$$gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l := gr_W Lie\mathcal{U}(\mathcal{O}_{K,\mathcal{S}}; l)/\langle \mathfrak{l} \mid l\rangle_{K,S} .$$

The Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ is also graded, i.e.

$$gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l = \bigoplus_{i=1}^{\infty}(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)_i.$$

We shall show that it has a correct number of generators.

We define a dual of $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ setting

$$(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond := \oplus_{i=1}^{\infty}(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)_i^*.$$

The vector space $(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$ is an l-adic analog of the generators of the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathcal{O}_{K,S}$.

In [9] we have studied the action of $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,-1,\infty\}; \overrightarrow{01})$. After standard embedding of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,-1,\infty\}; \overrightarrow{01})$ into the $\mathbb{Q}_l$-algebra $\mathbb{Q}_l\{\{X, Y_0, Y_1\}\}$ and passing to associated graded Lie algebra we get a Lie algebra representation

$$\Phi_{\overrightarrow{01}} : gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}], l) \longrightarrow Der^* Lie(X, Y_0, Y_1),$$

where $Der^* Lie(X, Y_0, Y_1)$ is a Lie algebra of special derivations of a free Lie algebra $Lie(X, Y_0, Y_1)$. The Lie ideal $\langle \mathfrak{l} \mid l\rangle_{\mathbb{Q},(2)}$ is contained in the kernel of $\Phi_{\overrightarrow{01}}$. Hence we get a morphism

$$\Phi_{\overrightarrow{01}} : gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l \to Der^* Lie(X, Y_0, Y_1).$$

Theorem 15.5.3 from [9] can be interpreted in the following way.

Theorem A. The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l)^\diamond$ is generated by the coefficients of the representation $\Phi_{\overrightarrow{01}}$.

We shall show that the natural map

$$gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l \longrightarrow gr_W Lie\mathcal{U}(\mathbb{Z})_l,$$

induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}]$, is a surjective morphism of Lie algebras. Let $I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})$ be its kernel. We say that $f \in (gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l)^\diamond$ is unramified everywhere if $f(I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})) = 0$. Our next result is then the immediate consequence of Theorem A.

Corollary B. The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z})_l)^\diamond$ is generated by these linear combinations of coefficients of the representation $\Phi_{\overrightarrow{01}}$, which are unramified everywhere.

The result mentioned at the beginning of the section is the Hodge-de Rham analogous of Corollary B.

We shall also consider the following situation. Let L be a finite Galois extension of K. We assume that a pair (V_L, v) or a triple (V_L, z, v) is defined over L. Then we get a representation of G_L on $\pi_1(V_{\bar{L}}; v)$ or $\pi(V_{\bar{L}}; z, v)$. We shall define what it means that a coefficient of a such representation is defined over K.

Then, working in Hodge-de Rham realization and assuming motivic formalism, one can show that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\,\mathbb{Z}[\frac{1}{3}]$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus (\{0, \infty\} \cup \mu_3)$ in one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$, $\frac{dz}{z-\xi_3}$, $\frac{dz}{z-\xi_3^2}$ ($\xi_3 = e^{\frac{2\pi i}{3}}$) from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are defined over $\mathbb{Q}$. However in this paper we shall show only an l-adic analog of that result.

Remark. A pair $(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{03})$ ramifies only at (3), hence periods of a mixed Tate motive associated with $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; \overrightarrow{03})$ are also periods of mixed Tate motives over $Spec\,\mathbb{Z}[\frac{1}{3}]$, but we do not know if in this way we shall get all such periods.

The final aim is to show that the vector space $(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^{\diamond}$ is generated by linear combinations of coefficients, which are unramified outside S and defined over K of representations of G_L for various L finite Galois extensions of K on fundamental groups or on torsors of paths of a projective line minus a finite number of points or perhaps some other varieties. This will imply (by the very definition) that all mixed Tate representations of $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ are of geometric origin. We are however very far from this aim.

Then we must pass from Lie algebra representations of $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ to the representation of the corresponding group in order to show that any mixed Tate representation of G_K is of geometric origin. This part of the problem is not studied here.

The results of this paper where presented in a seminar talk in Lille in May 2009 and then at the end of my lectures at the summer school at Galatasaray University in Istanbul in June 2009.

While finishing this paper the author has a delegation in CNRS in Lille at the *Laboratoire*, Paul Painlevé and he would like to thank very much the director, Professor Jean D'Almeida for accepting him in the Painleve Laboratory. Thanks are also due to Professor Douai who helped me to get this delegation.

In [11] we were studying related questions. In the sequel we make some comments concerning [11], as the results presented there are not complete.

1 Weighted Tate completions of Galois groups

Let K be a number field and let S a finite set of finite places of K. Let $\mathcal{O}_{K,S}$ be the ring of S-integers in K, i.e.

$$\mathcal{O}_{K,S} := \{\frac{a}{b} \mid a, b \in \mathcal{O}_K,\ b \notin \mathfrak{p} \text{ for all } \mathfrak{p} \notin S\}.$$

Let us fix a rational prime l. We denote by $\{\mathfrak{l} \mid l\}_K$ a set of finite places of K lying over a prime ideal (l) of $\mathbb{Z}$.

Let $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ be the weighted Tate $\mathbb{Q}_l$-completion of the étale fundamental group $\pi_1^t(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K})$. The group $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is an affine, pro-algebraic group over $\mathbb{Q}_l$ equipped with the homomorphism

$$\pi_1^t(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K}) \longrightarrow \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)(\mathbb{Q}_l)$$

with a Zariski dense image, such that any weighted Tate finite dimensional $\mathbb{Q}_l$-representation of $\pi_1^t(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K})$ factors through $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$. We point out that weighted Tate finite dimensional $\mathbb{Q}_l$-representations of $\pi_1^t(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K})$ correspond to representations of G_K unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$.

There is an exact sequence

$$1 \to \mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathbb{G}_m \to 1.$$

The kernel $\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is a prounipotent affine group over $\mathbb{Q}_l$ equipped with the weight filtration $\{W_{-2i}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)\}_{i\in\mathbb{N}}$. The associated graded Lie algebra

$$gr_W\, Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) := \bigoplus_{i=1}^{\infty} gr_W\, Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_i,$$

where

$$gr_W\, Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_i := W_{-2i}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/W_{-2(i+1)}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l),$$

is a free Lie algebra.

In degree 1 there are functorial isomorphisms

(1.1.a.) $$gr_W\, Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1 \approx Hom(\mathcal{O}^*_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathbb{Q}_l).$$

and

(1.1.b.)
$$(gr_W\, Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1)^* \approx \mathcal{O}^*_{K,S\cup\{\mathfrak{l}|l\}_K} \otimes \mathbb{Q}_l \approx H^1(Spec\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathbb{Q}_l(1)).$$

In degree $i > 1$ there are functorial isomorphisms

(1.1.c.)
$$\big(gr_W\, Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/\Gamma^2 gr_W\, Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)\big)_i^* \approx H^1(G_K; \mathbb{Q}_l(i)).$$

(see [5], [6]).

Let us assume that a pair (V, v) is defined over K and has good reduction outside S. Then the representation of G_K on pro-l quotient of $\pi_1^t(V_{\bar{K}}; v)$ is unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$ and is ramified at all finite places of K which lie over (l). This has an effect that the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ has more generators in degree 1 than the corresponding Lie algebra of the tannakian category of mixed Tate motives over Spec$\mathcal{O}_{K,S}$.

We shall show below how to kill these additional generators corresponding to finite places of K lying over (l), which are not in S.

Let $u \in \mathcal{O}^*_{K,S\cup\{\mathfrak{l}|l\}_K}$ and let $\kappa(u) : G_K \to \mathbb{Z}_l$ be the Kummer character of u. The representation

$$G_K \ni \sigma \longrightarrow \begin{pmatrix} 1 & 0 \\ \kappa(u)(\sigma) & \chi(\sigma) \end{pmatrix} \in GL_2(\mathbb{Q}_l)$$

is an l-adic weighted Tate representation of G_K unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$, i.e. it is an l-adic weighted Tate representation of $\pi_1^t(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; Spec\bar{K})$. Hence the Kummer character $\kappa(u)$ we can view also as a homomorphism

$$\kappa(u) : gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1 \to \mathbb{Q}_l,$$

by 1.1.b. Let us set

$$(\mathfrak{l} \mid l)_{K,S} := \bigcap_{u \in \mathcal{O}^*_{K,S}} \big(Ker(\kappa(u) : grLie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1 \to \mathbb{Q}_l)\big)$$

and let $\langle \mathfrak{l} \mid l \rangle_{K,S}$ be a Lie ideal of $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ generated by elements of $(\mathfrak{l} \mid l)_{K,S}$.

Proposition 1.2.

i) The quotient Lie algebra

$$gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/\langle \mathfrak{l} \mid l \rangle_{K,S}$$

is graded.

ii) The quotient Lie algebra $grLie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/\langle \mathfrak{l} \mid l \rangle_{K,S}$ is free, freely generated by $n_1 = \dim_{\mathbb{Q}}(\mathcal{O}^*_{K,S} \otimes \mathbb{Q})$ elements in degree 1, and by $n_i = \dim_{\mathbb{Q}_l}(H^1(G_K; \mathbb{Q}_l(i))$ elements in degree $i > 1$.

iii) Let $a_1, \ldots, a_{n+1}$ be K-points of $\mathbb{P}^1_K$ and let $V := \mathbb{P}^1_K \setminus \{a_1, \ldots, a_{n+1}\}$. Let z and v be K-points of V or tangential points defined over K. Let us assume that a pair (V, v) (resp. a triple (V, z, v)) has good reduction outside S. Then the morphism of Lie algebras

$$gr_W Lie\varphi_{V,v} : gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to Der^* Lie(X_1, \ldots, X_n)$$

resp.

$$gr_W Lie\psi_{V,z,v} : gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to Lie(X_1, \dots, X_n) \tilde{\times} Der^* Lie(X_1, \dots, X_n)$$

deduced from the action of G_K on $\pi_1(V_{\bar{K}}; v)$ (resp. on $\pi(V_{\bar{K}}; z, v)$) factors through the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/\langle \mathfrak{l} \mid l\rangle_{K,S}$.

Proof. The Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$ of the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is generated by elements of degree 1, hence it is homogenous. Therefore the quotient Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/\langle \mathfrak{l} \mid l\rangle_{K,S}$ has a natural grading induced from that of $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$.

Let us choose $u_1, \dots, u_p \in \mathcal{O}^*_{K,S}$ ($p = \dim \mathcal{O}^*_{K,S} \otimes \mathbb{Q}$) such that $u_1 \otimes 1, \dots, u_p \otimes 1$ is a base of $\mathcal{O}^*_{K,S} \otimes \mathbb{Q}$. Let $z_1, \dots, z_q \in \mathcal{O}^*_{K,S\cup\{\mathfrak{l}|l\}_K}$ be such that $u_1 \otimes 1, \dots, u_p \otimes 1, z_1 \otimes 1, \dots, z_q \otimes 1$ is a base of $(\mathcal{O}^*_{K,S\cup\{\mathfrak{l}|l\}_K}) \otimes \mathbb{Q}$. Let $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$ be a base of $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1$ dual to the Kummer characters $\kappa(u_1), \dots, \kappa(u_p)$, $\kappa(z_1), \dots, \kappa(z_q)$. Then $\beta_1, \dots, \beta_q$ generate the Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$. The point ii) follows now immediately from the fact that the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is free, freely generated by elements $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$ in degree 1 and by n_i generators in degrees $i > 1$ (see [5] and [6]).

Let us assume that a pair (V, v) (resp. a triple (V, z, v)) has good reduction outside S. We shall show in the next lemma that then the morphism $gr_W Lie\varphi_{V,v}$ (resp. $gr_W Lie\psi_{V,z,v}$) in degree 1 is given by Kummer characters of elements belonging to $\mathcal{O}^*_{K,S}$. This implies that the morphism vanishes on $(\mathfrak{l} \mid l)_{K,S}$, hence it vanishes on $\langle \mathfrak{l} \mid l\rangle_{K,S}$. Hence the point iii) follows immediately. □

Lemma 1.2.1. Let us assume that a pair (V, v) (resp. a triple (V, z, v)) has good reduction outside S. Then the morphism $gr_W Lie\varphi_{V,v}$ (resp. $gr_W Lie\psi_{V,z,v}$) in degree 1 is given by Kummer characters of elements belonging to $\mathcal{O}^*_{K,S}$.

Proof. For simplicity we shall consider only a pair (V, v), where v is a K-point. The definition of good reduction at a finite place $\mathfrak{p}$ depends only on an isomorphism class of (V, v) over K (see [10], definition 17.5), hence we can assume that $a_1 = 0$, $a_2 = 1$ and $a_{n+1} = \infty$.

The morphism $gr_W Lie\varphi_{V,v}$ is given in degree 1 by Kummer characters $\kappa(\frac{a_i - a_k}{v - a_k})$ for $i \neq k$ and $i, k \in \{1, 2, \dots, n\}$ (see [10], 17.10.a). Let $\mathcal{S}(V, v)$ be a set of finite places $\mathfrak{p}$ of K such that there exists a pair (i, k) satisfying $i \neq k$ and such that $\mathfrak{p}$ valuation of $\frac{a_i - a_k}{v - a_k}$ is different from 0. Then clearly $\frac{a_i - a_k}{v - a_k} \in \mathcal{O}^*_{K,\mathcal{S}(V,v)}$ for all pair (i, k) with $i \neq k$.

For the pair (V, v) the notion of good reduction at $\mathfrak{p}$ and strong good reduction at $\mathfrak{p}$ coincide (see [10], Definitions 17.4, 17.5 and Corollary 17.18). It follows from Lemma 17.15 in [10] that $\mathfrak{p} \notin S$ implies $\mathfrak{p} \notin \mathcal{S}(V, v)$. Hence $\mathcal{S}(V, v) \subset S$. Therefore $\frac{a_i - a_k}{v - a_k} \in \mathcal{O}^*_{K,S}$ for all pairs (i, k) with $i \neq k$. □

Definition 1.3. We set

$$gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l := gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/\langle \mathfrak{l} \mid l\rangle_{K,S}.$$

Let S and S_1 be finite disjoint sets of finite places of K. The inclusion of rings

$$\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K} \hookrightarrow \mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}$$

induces morphisms of weighted Tate $\mathbb{Q}_l$-completions

$$\pi^{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}_{K,S\cup\{\mathfrak{l}|l\}_K} : \mathcal{G}(\mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};l) \longrightarrow \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$$

and

$$\pi^{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}_{K,S\cup\{\mathfrak{l}|l\}_K} : \mathcal{U}(\mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};l) \longrightarrow \mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l).$$

After passing to associated graded Lie algebras with respect to weight filtrations one gets a morphism of associated graded Lie algebras

$$gr_W Lie\pi^{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}_{K,S\cup\{\mathfrak{l}|l\}_K} : gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};l) \longrightarrow gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l).$$

Proposition 1.4. Let S and S_1 be finite disjoint sets of finite places of K. The inclusion of rings $\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K} \hookrightarrow \mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}$ induces a morphism of graded Lie algebras

$$(gr_W Lie\pi^{K,S\cup S_1}_{K,S})_l : gr_W Lie(\mathcal{O}_{K,S\cup S_1})_l \to gr_W Lie(\mathcal{O}_{K,S})_l.$$

Proof. The inclusion of rings

$$i : \mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K} \hookrightarrow \mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}$$

induces a morphism of associated graded Lie algebras

$$gr_W Lie\pi^{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}_{K,S\cup\{\mathfrak{l}|l\}_K} : gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};l) \longrightarrow gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l).$$

To simplify the notation we denote by π the degree 1 component of the morphism $gr_W Lie\pi^{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}_{K,S\cup\{\mathfrak{l}|l\}_K}$. In degree 1 we have a commutative diagram

$$\begin{array}{ccc} gr_W Lie(\mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};l)_1 & \xrightarrow{\pi} & gr_W Lie((\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1 \\ \approx\downarrow & & \approx\downarrow \\ Hom(\mathcal{O}^*_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};\mathbb{Q}_l) & \longrightarrow & Hom(\mathcal{O}^*_{K,S\cup\{\mathfrak{l}|l\}_K};\mathbb{Q}_l) \end{array}$$

by (1.1.a.). Let $z \in gr_W Lie(\mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};l)_1$ be such that $\kappa(u)(z) = 0$ for all $u \in \mathcal{O}^*_{K,S\cup S_1}$. Then $\kappa(u)(\pi(z)) = 0$ for all $u \in \mathcal{O}^*_{K,S}$ because of the commutativity of the diagram.

Hence π maps $(\mathfrak{l} \mid l)_{K,S\cup S_1}$ into $(\mathfrak{l} \mid l)_{K,S}$. This implies that the morphism $gr_W Lie\pi^{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}_{K,S\cup\{\mathfrak{l}|l\}_K} : gr_W Lie(\mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};l) \to gr_W Lie(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ induces a morphism of graded Lie algebras

$$(gr_W Lie\pi^{K,S\cup S_1}_{K,S})_l : gr_W Lie(\mathcal{O}_{K,S\cup S_1})_l \to gr_W Lie(\mathcal{O}_{K,S})_l.$$

□

It follows from (1.1.a) and from the construction of the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ that the degree 1 part of the morphism from Proposition 1.4 inserts into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Hom(\mathcal{O}^*_{K,S\cup S_1}/\mathcal{O}^*_{K,S};\mathbb{Q}_l) & \longrightarrow & Hom(\mathcal{O}^*_{K,S\cup S_1};\mathbb{Q}_l) & \longrightarrow & Hom(\mathcal{O}^*_{K,S};\mathbb{Q}_l) & \to & 0 \\ & & \approx\downarrow & & \approx\downarrow & & \approx\downarrow & & \\ 1 & \to & (ker(gr_W Lie\pi^{K,S\cup S_1}_{K,S})_l)_1 & \longrightarrow & (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1})_l)_1 & \longrightarrow & (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)_1 & \to & 0\,. \end{array}$$

Definition 1.5. Let $I(\mathcal{O}_{K,S\cup S_1}:\mathcal{O}_{K,S})$ be a Lie ideal of a Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1})_l$ generated by $(ker(gr_W Lie\pi^{K,S\cup S_1}_{K,S})_l)_1$.

Proposition 1.6. The morphism $(gr_W Lie\pi^{K,S\cup S_1}_{K,S})_l$ induces an isomorphism of graded Lie algebras

$$\bar{\pi}^{K,S\cup S_1}_{K,S} : gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1})_l/I(\mathcal{O}_{K,S\cup S_1}:\mathcal{O}_{K,S}) \longrightarrow gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l.$$

Proof. The Lie ideal $I(\mathcal{O}_{K,S\cup S_1}:\mathcal{O}_{K,S})$ is generated by elements of degree 1. Hence the quotient Lie algebra carries a natural induced grading and the morphism $\bar{\pi}^{K,S\cup S_1}_{K,S}$ is a morphism of graded Lie algebras.

Moreover free generators of the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K};l)$ in degree k greater than 1 are mapped bijectively onto free generators in degree k of the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ by the morphism $gr_W Lie\pi^{K,S\cup S_1\cup\{\mathfrak{l}|l\}_K}_{K,S\cup\{\mathfrak{l}|l\}_K}$ by (1.1.c.). Hence the same is true for the morphism

$$(gr_W Lie\pi^{K,S\cup S_1}_{K,S})_l : gr_W Lie(\mathcal{O}_{K,S\cup S_1})_l \to gr_W Lie(\mathcal{O}_{K,S})_l.$$

This implies that the induced morphism $\bar{\pi}^{K,S\cup S_1}_{K,S}$ is an isomorphism of graded Lie algebras. □

Definition 1.7. Let $L=\bigoplus_{i=1}^{\infty} L_i$ be a graded Lie algebra over a field k such that $\dim L_i<\infty$ for every i. We set

$$L^\diamond := \bigoplus_{i=1}^{\infty} Hom(L_i,k)$$

and we call $L^\diamond$ a dual of L.

The Lie bracket $[\,,\,]$ of the Lie algebra L induces a morphism

$$d := [\,,\,]^* : L^\diamond \to L^\diamond \otimes L^\diamond,$$

whose image is contained in the subspace of $L^\diamond\otimes L^\diamond$ generated by antisymmetrical tensors of the form $a\otimes b - b\otimes a$.

Definition 1.8. The $\mathbb{Q}_l$-vector space $(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$ we shall call a vector space of coefficients on $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$.

Remark 1.8.1. We consider the $\mathbb{Q}_l$-vector space $(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$ as an analog of generators of the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $Spec\mathcal{O}_{K,S}$. Anticipating, if the vector space $(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$ is generated by geometric coefficients then it has a natural $\mathbb{Q}$-structure, which is much better analog.

The morphism

$$d = [\,,\,]^* : (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond \to (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond \otimes (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$$

we denote by $d_{\mathcal{O}_{K,S}}$. We set

$$\mathcal{L}(\mathcal{O}_{K,S};l) := ker(d_{\mathcal{O}_{K,S}}).$$

It follows from from the construction of the graded Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ and from isomorphisms (1.1.a.), (1.1.b.) and (1.1.c.) that there are natural isomorphisms

$$\mathcal{L}(\mathcal{O}_{K,S};l)_i = ker(d_{\mathcal{O}_{K,S}})_i \approx H^1(G_K;\mathbb{Q}_l(i)) \quad \text{for} \quad i>1$$

and

$$\mathcal{L}(\mathcal{O}_{K,S};l)_1 = ker(d_{\mathcal{O}_{K,S}})_1 = (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond_1 \approx \mathcal{O}^*_{K,S} \otimes \mathbb{Q}_l.$$

The surjective morphism

$$(gr_W Lie\pi^{K,S\cup S_1}_{K,S})_l : gr_W Lie(\mathcal{O}_{K,S\cup S_1})_l \to gr_W Lie(\mathcal{O}_{K,S})_l$$

induces a monomorphism of dual vector spaces

$$\Pi^{K,S}_{K,S\cup S_1} := \big((gr_W Lie\pi^{K,S\cup S_1}_{K,S})_l\big)^\diamond : (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond \longrightarrow (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1})_l)^\diamond.$$

Corollary 1.9. We have

$$(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond \approx \{f \in (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1})_l)^\diamond \mid f(I(\mathcal{O}_{K,S\cup S_1} : \mathcal{O}_{K,S}))=0\}.$$

Proof. The corollary follows directly from Proposition 1.6. □

Definition 1.10. Let S and S_1 be finite disjoint sets of finite places of K. We say that

$$f \in (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup S_1})_l)^\diamond$$

is unramified outside S_1 if $f(I(\mathcal{O}_{K,S\cup S_1} : \mathcal{O}_{K,S}))=0$.

Now we shall study relations between weighted Tate completions of a Galois group of a field and a Galois group of its finite Galois extention.

Let K be a number field. Let S be a finite set of finite places of K. Let L be a finite Galois extension of K and let T be a set of finite places of L lying over finite places of K belonging to S. The inclusion of rings of algebraic integers

$$\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K} \hookrightarrow \mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}$$

induces morphisms of weighted Tate completions

$$\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K} : \mathcal{G}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \to \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$$

and

$$\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K} : \mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \to \mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l).$$

After passing to associated graded Lie algebras with respect to weight filtrations we get morphisms of graded Lie algebras

1.10.1.

$$gr_W Lie\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K} : gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \to gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l).$$

Proposition 1.11. Let K be a number field and let L be a finite Galois extension of K. Let S be a set of finite places of K and let T be a set of finite places of L lying over elements of S. The inclusion of rings of algebraic integers $\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K} \hookrightarrow \mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}$ induces a morphism of graded Lie algebras

$$(gr_W Lie\pi^{L,T}_{K,S})_l : gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l \to gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l.$$

Proof. Let $u \in \mathcal{O}^*_{K,S}$. Then $u \in \mathcal{O}^*_{L,T}$. Hence the Kummer character $\kappa(u)$ considered on $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)_1$ vanishes on a subspace $(\mathfrak{l} \mid l)_{L,T}$ of $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)_1$. But this implies that the morphism 1.10.1 maps the Lie ideal $\langle \mathfrak{l} \mid l\rangle_{L,T}$ into the Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$. Passing to quotient Lie algebras we get a morphism

$$(gr_W Lie\pi^{L,T}_{K,S})_l : gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l \to gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l.$$

. □

Proposition 1.12. Let L be a finite Galois extension of K. Let S be a set of finite places of K and let T be a set of finite places of L lying over elements of S. Then the morphisms

$$gr_W Lie\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K} : gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \to gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$$

and

$$(gr_W Lie\pi^{L,T}_{K,S})_l : gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l \to gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$$

are surjective.

Proof. It follows from (1.1.b) and (1.1.c) that the morphism 1.10.1 induces the following commutative diagram

$$\begin{array}{ccc} Hom((gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)/\Gamma^2..)_i;\mathbb{Q}_l) & \longrightarrow & Hom((gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)/\Gamma^2..)_i;\mathbb{Q}_l) \\ \approx\uparrow & & \approx\uparrow \\ H^1(Spec\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};\mathbb{Q}_l(i)) & \longrightarrow & H^1(Spec\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};\mathbb{Q}_l(i)) \end{array}$$

for each $i > 0$. The map of cohomology groups induced by the inclusion $\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K} \hookrightarrow \mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}$ is injective. Hence the morphism

$$gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)/\Gamma^2... \longrightarrow gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)/\Gamma^2...$$

is surjective. This implies that the morphism

$$gr_W Lie\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K} : gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \longrightarrow gr_W Lie\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$$

is surjective. Hence the morphism $(gr_W Lie\pi^{L,T}_{K,S})_l$ is also surjective. □

Definition 1.13. Let us set

$$I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S}) := ker\big((gr_W Lie\pi^{L,T}_{K,S})_l : gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l \to gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l\big).$$

Let us set $G = Gal(L/K)$. From now on we shall assume that

i) l does not divide the order of G;

ii) $K(\mu_{l^\infty}) \cap L = K$.

Then it follows from [11], Lemma 4.2.2 and the functoriality of the weighted Tate completion that the group G acts on $\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)$ preserving weight filtration.

Lemma 1.14. The action of G on $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)$ induces an action of G on the graded Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l$.

Proof. We recall first how G acts on $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)$. Let M be a maximal pro-l unramified outside $T \cup \{\mathfrak{l} \mid l\}_L$ extension of $L(\mu_{l^\infty})$. Then $G \subset Gal(M/K(\mu_{l^\infty}))$. Hence G acts on $Gal(M/L(\mu_{l^\infty}))$ by conjugation (see [4], section 1 or [11], Lemma 4.2.2.). Then by functoriality G acts also on $\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)$ preserving weight filtration. Therefore G acts also on $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)$ by Lie algebra automorphisms.

For any $a \in \mathcal{O}^*_{L,T}$, any $\sigma \in Gal(M/L(\mu_{l^\infty}))$ and any $g \in G$ we have

$$\kappa(g^{-1}(a))(\sigma) = \kappa(a)(g \cdot \sigma \cdot g^{-1}).$$

This implies that for any $s \in gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)_1$ we have

$$\kappa(g^{-1}(a))(s) = \kappa(a)(g(s)).$$

Hence the subspace $(\mathfrak{l} \mid l)_{L,T}$ of $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; l)_1$ is G-invariant, i.e. $G((\mathfrak{l} \mid l)_{L,T}) \subset (\mathfrak{l} \mid l)_{L,T}$. Therefore G acts also on the quotient Lie algebra. □

Proposition 1.15. Let L be a finite Galois extension of K. Let S be a finite set of finite places of K and let T be a set of finite places of L lying over elements of S. We also assume that

i) l does not divide the order of G;

ii) $K(\mu_{l^\infty}) \cap L = K$.

Then we have:

i) the Lie ideal $I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$ is generated by elements $gv - v$, where $g \in G$ and $v \in gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l$;

ii) The morphism $(gr_W Lie\pi^{L,T}_{K,S})_l$ induces an isomorphism

$$gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l / I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S}) \longrightarrow gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l.$$

Proof. The group G acts on $\mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}; l)$ by automorphisms preserving the weight filtration. Lemma 1.14 implies that it acts by automorphisms on the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l$. Then $\Gamma^2 gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l$ is a G-invariant subspace of $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l$. Let us choose in each degree i a G-invariant complement V_i to $(\Gamma^2 gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l)_i$. Then the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l$ is generated by a vector space $\oplus_{i=1}^{\infty} V_i$.

Let V_i' be the image of V_i in $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ by the map $(gr_W Lie\pi^{L,T}_{K,S})_l$. We have isomorphisms compatible with the action of the group G

$$V_1^* = (gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l)_1^* \approx \mathcal{O}_{L,T}^* \otimes \mathbb{Q}_l \ \text{ for } \ i = 1$$

and

$$V_i^* \approx \big(gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l / \Gamma^2 gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l\big)_i^* \approx H^1(Spec\mathcal{O}_{L,T}; \mathbb{Q}(i))$$

for $i > 1$. Hence it follows that in degree 1 there are isomorphisms

$$\begin{array}{ccccccc} ((V_1)_G)^* & \approx & (V_1^*)^G & \approx & \big((gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l)_1^*\big)^G & \approx & (\mathcal{O}_{L,T}^* \otimes \mathbb{Q}_l)^G \\ & & & & \approx\uparrow & & \approx\uparrow \\ & & & & (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)_1^* & \approx & \mathcal{O}_{K,S}^* \otimes \mathbb{Q}_l. \end{array}$$

For $i > 1$ we have a commutative diagram

$$\begin{array}{ccc}
Hom((gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l/\Gamma^2..)_i;\mathbb{Q}_l) & \longrightarrow & Hom((gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l/\Gamma^2..)_i;\mathbb{Q}_l) \\
=\Big\uparrow & & =\Big\uparrow \\
(ker d_{\mathcal{O}_{K,S}})_i & \longrightarrow & (ker d_{\mathcal{O}_{L,T}})_i \\
\approx\Big\uparrow & & \approx\Big\uparrow \\
H^1(Spec\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};\mathbb{Q}_l(i)) & \longrightarrow & H^1(Spec\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};\mathbb{Q}_l(i)).
\end{array}$$

The isomorphism

$$H^1(Spec\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};\mathbb{Q}(i)) \approx H^1(Spec\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};\mathbb{Q}(i))^G,$$

which follows from the Lyndon spectral sequence and the isomorphism

$$H^1(Spec\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};\mathbb{Q}(i))^G = (V_i^*)^G = ((V_i)_G)^*$$

imply that

$$V_i' \approx V_i/R_i,$$

where R_i is a subspace of V_i generated by elements $gv - v$ for $g \in G$ and $v \in V_i$.

Observe that $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ is generated freely by a base of $\oplus_{i=1}^{\infty} V_i'$. Therefore the Lie ideal generated by $\oplus_{i=1}^{\infty} R_i$ is contained in $I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$. But the quotient of $gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l$ by the ideal generated by $\oplus_{i=1}^{\infty} R_i$ is isomorphic to $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$. Hence we get i) and therefore also ii). □

Corollary 1.16. We have

$$(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond \approx \{f \in (gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l)^\diamond \mid f(I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})) = 0\}.$$

Definition 1.17. We say that $f \in (gr_W Lie\mathcal{U}(\mathcal{O}_{L,T})_l)^\diamond$ is defined over K or that f is $[Gal(L/K)]$-invariant if $f(I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})) = 0$.

We finish this section with the study of the dual of the Lie bracket of the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$.

The Lie bracket of the Lie algebra $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ induces

$$d_{\mathcal{O}_{K,S}} : (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond \longrightarrow (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond \otimes (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond.$$

We recall that we have defined

$$\mathcal{L}(\mathcal{O}_{K,S}; l) := ker\ d_{\mathcal{O}_{K,S}}.$$

The vector space $\mathcal{L}(\mathcal{O}_{K,S};l)$ inherits grading from $(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$ and we have

$$\mathcal{L}(\mathcal{O}_{K,S};l) = \bigoplus_{i=1}^{\infty} \mathcal{L}(\mathcal{O}_{K,S};l)_i.$$

Observe that

$$\mathcal{L}(\mathcal{O}_{K,S};l) = \{f \in (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond \ \mid \ f(\Gamma^2 gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l) = 0\}$$
$$= ((gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^{ab})^\diamond.$$

To simplify the notation we denote $d_{\mathcal{O}_{K,S}}$ by d and we set

$$gu := (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond.$$

We define operators

$$d^{(n)} : (gu)^\diamond \longrightarrow \bigotimes_{i=1}^{n+1} (gu)^\diamond$$

by setting

$$d^{(n)} = (d \otimes (\otimes_{i=1}^{n-1} Id_{(gu)^\diamond}) \circ \ldots \circ (d \otimes Id_{(gu)^\diamond} \otimes Id_{(gu)^\diamond}) \circ (d \otimes Id_{(gu)^\diamond}) \circ d.$$

Let $pr_{n+1} : \otimes_{i=1}^{n+1} gu \longrightarrow gu$ be given by

$$pr_{n+1}(u_1 \otimes u_2 \otimes \ldots \otimes u_{n+1}) := [[\ldots[[u_1, u_2], u_3], \ldots], u_{n+1}]$$

Lemma 1.18. We have

i) $(pr_{n+1})^\diamond = d^{(n)}$;

ii) $f \in (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$ vanishes on $\Gamma^{n+1}(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)$ if and only if $d^{(n)}(f) = 0$;

iii) Let $f \in (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond_m$. Assume that $d^{(k+1)}(f) = 0$. Then $d^{(k)}(f) \in \bigotimes_{i=1}^{k+1} \mathcal{L}(\mathcal{O}_{K,S};l)$.

Proof. The point i) is clear and ii) follows from i).

It rests to show the point iii). The map $d^{(k)}(f) = f \circ pr_{k+1}$ is equal to the composition

$$\otimes_{i=1}^{k+1} gu \to \otimes_{i=1}^{k+1} (gu)^{ab} \to \Gamma^{k+1} gu / \Gamma^{k+2} gu \hookrightarrow gu/\Gamma^{k+2} gu \xrightarrow{f} \mathbb{Q}_l.$$

The equality $\mathcal{L}(\mathcal{O}_{K,S};l) = ((gu)^{ab})^\diamond$ implies that $d^{(k)}(f) \in \bigotimes_{i=1}^{k+1} \mathcal{L}(\mathcal{O}_{K,S};l)$. $\square$

2 Geometric coefficients

Let $a_1,\dots,a_n \in K$ and let $V := \mathbb{P}^1_K \setminus \{a_1,\dots,a_n,\infty\}$. Let v and z be K-points of V or tangential points defined over K. Let S be a finite set of finite places of K. Let l be a fixed rational prime.

We denote by $\pi_1(V_{\bar K};v)$ the pro-l completion of the étale fundamental group of $V_{\bar K}$ based at v and by $\pi(V_{\bar K};z,v)$ the $\pi_1(V_{\bar K};v)$-torseur of pro-l paths from v to z.

The Galois group G_K acts on $\pi_1(V_{\bar K};v)$ and on $\pi(V_{\bar K};z,v)$. After the standard embedding of $\pi_1(V_{\bar K};v)$ into a $\mathbb{Q}_l$-algebra $\mathbb{Q}_l\{\{X_1,\dots,X_n\}\}$ of formal power series in non-commuting variables we get two Galois representations

$$\varphi_v = \varphi_{V,v} : G_K \longrightarrow Aut(\mathbb{Q}_l\{\{X_1,\dots,X_n\}\})$$

and

$$\psi_{z,v} = \psi_{V;z,v} : G_K \longrightarrow GL(\mathbb{Q}_l\{\{X_1,\dots,X_n\}\})$$

deduced from actions of G_K on π_1 and on the π_1-torseur (see [7], section 4).

Let us assume that a pair (V,v) and a triple (V,z,v) are unramified outside S. Then the representations $\varphi_{V,v}$ and $\psi_{V,z,v}$ factor through the weighted Tate $\mathbb{Q}_l$-completion $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ of $\pi_1^t(Spec\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};Spec\bar K)$ because the representations $\varphi_{V,v}$ and $\psi_{V,z,v}$ are l-adic weighted Tate representations (see [11] Proposition 1.0.3). It follows from Proposition 1.2 iii) that passing to associated graded Lie algebras with respect to the weight filtrations we get morphisms of graded Lie algebras

$$gr_W Lie\varphi_v : gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l \to (\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle;\{\,\})$$

and

$$gr_W Lie\psi_{z,v} : gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l \to Lie(X_1,\dots,X_n)\tilde\times(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle;\{\})$$

Passing to dual vector spaces we get morphisms

$$\Phi^v := (gr_W Lie\varphi_v)^\diamond : (\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle;\{\,\})^\diamond \to (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$$

and

$$\Psi^{z,v} := (gr_W Lie\psi_{z,v})^\diamond : (Lie(X_1,\dots,X_n)\tilde\times(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle;\{\,\}))^\diamond \to (gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond.$$

Definition 2.1. We set

$$Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,v) := Image\,(\Phi^v)$$

and

$$Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,z,v) := Image\,(\Psi^{z,v}).$$

The vector subspace $Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,v)$ (resp. $Geom\,Coeff^l_{\mathcal{O}_{K,S}}(V,z,v)$) of $(gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l)^\diamond$ we shall call a vector space of geometric coefficients on $gr_W Lie\mathcal{U}(\mathcal{O}_{K,S})_l$ coming from (V,v) (resp. (V,z,v)).

Let us fix a Hall base $\mathcal{B}$ of a free Lie algebra $Lie(X_1,\dots,X_n)$. If $e\in\mathcal{B}$ then e^* denotes a dual linear form in $Lie(X_1,\dots,X_n)^\diamond$. Let

$$pr_{i_0} : \bigoplus_{i=1}^{n} Lie(X_1,\dots,X_n)/\langle X_i\rangle \longrightarrow Lie(X_1,\dots,X_n)/\langle X_{i_0}\rangle$$

be the projection on the i_0-th component. Let

$$p : Lie(X_1,\dots,X_n)\tilde{\times}(\bigoplus_{i=1}^{n} Lie(X_1,\dots,X_n)/\langle X_i\rangle) \longrightarrow Lie(X_1,\dots,X_n)$$

be the projection on the first factor.

We set

2.2. $$\{z,v\}_{e^*} := e^*\circ p\circ gr_W Lie\psi_{z,v} = \Psi^{z,v}(e^*\circ p).$$

Let $e\in\mathcal{B}$ be different from X_i. Let $\overrightarrow{a_i}$ be any tangential point defined over K at a_i. Then we have

2.3. $$\{\overrightarrow{a_i},v\}_{e^*} = \Phi^v(e^*\circ pr_i) = e^*\circ pr_i\circ gr_W Lie\varphi_v.$$

The geometric coefficients $\{z,v\}_{e^*}$ considered here are the l-adic iterated integrals from [7]. We use here the notation $\{z,v\}_{e^*}$ because it is more convenient for our study.

If $\psi \in (Lie(X_1,\dots,X_n)\tilde{\times}(\bigoplus_{i=1}^n Lie(X_1,\dots,X_n)/\langle X_i\rangle)^\diamond$ then $\Psi^{z,v}(\psi) = \psi\circ gr_W Lie\psi_{z,v}$ is a linear combination of symbols 2.2 and 2.3.

Let us assume that a triple (W,ζ,ξ) is unramified outside $S\cup S_1$. We would like to have some working criterium when elements of $Geom\,Coeff^l_{\mathcal{O}_{K,S\cup S_1}}(W,\zeta,\xi)$ are unramified outside S_1.

Let us assume that a triple (U,s,t) is defined over L. Similarly we would like to know when $f\in Geom\,Coeff^l_{\mathcal{O}_{L,T}}(U,s,t)$ is defined over K, i.e, when f is $[Gal(L/K)]$-invariant. We shall study some explicit examples in the next sections.

3 From $\mathbb{P}^1 \setminus \{0, 1, -1, \infty\}$ to periods of mixed Tate motives over Spec$\mathbb{Z}$

Let $V := \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, -1, \infty\}$. In [9], 15.5 we have studied the Galois representation

3.0. $$\varphi_{\vec{01}} : G_{\mathbb{Q}} \to Aut(\pi_1(V_{\bar{\mathbb{Q}}}; \vec{01})).$$

Observe that the pair $(V, \vec{01})$ has good reduction outside the prime ideal (2) of $\mathbb{Z}$ (see [11], Definition 2.0). Hence the representation 3.0 is unramified outside prime ideals (2) and (l) (see [10], Corollary 17.17). After the standard embedding of $\pi_1(V_{\bar{\mathbb{Q}}}; \vec{01})$ into the $\mathbb{Q}_l$-algebra of formal power series in non-commuting variables $\mathbb{Q}_l\{\{X, Y_0, Y_1\}\}$ (see [9], 15.2) we get a representation

3.1. $$\varphi_{\vec{01}} : G_{\mathbb{Q}} \to Aut(\mathbb{Q}_l\{\{X, Y_0, Y_1\}\}).$$

It follows from the universal properties of weighted Tate completion that the morphism 3.1 factors through

$$\varphi_{\vec{01}} : \mathcal{G}(\mathbb{Z}[\frac{1}{2l}]; l) \to Aut(\mathbb{Q}_l\{\{X, Y_0, Y_1\}\}).$$

Passing to associated graded Lie algebras we get a morphism of graded Lie algebras studied in [9], 15.5,

3.2. $$gr_W Lie\varphi_{\vec{01}} : gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2l}]; l) \to (Lie(X, Y_0, Y_1), \{\,\}).$$

It follows from Proposition 1.2 iii) that the morphism 3.2 induces a morphism of graded Lie algebras

3.3. $$(gr_W Lie\varphi_{\vec{01}})_l : gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l \to (Lie(X, Y_0, Y_1), \{\,\}).$$

Proposition 3.4. The morphism of graded Lie algebras

$$(gr_W Lie\varphi_{\vec{01}})_l : gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l \to (Lie(X, Y_0, Y_1), \{\,\})$$

deduced from the action of $G_{\bar{\mathbb{Q}}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, -1, \infty\}; \vec{01})$ is injective.

Proof. The proposition follows from [9], Theorem 15.5.3. Below we give a more detailed proof.

We recall that $\{G_i(V, \vec{01})\}_{i \in \mathbb{N}}$ is a filtration of $G_{\mathbb{Q}}$ associated with the representation 3.0 (see [7], section 3). The pair $(V, \vec{01})$ has good reduction outside the prime ideal (2) of $\mathbb{Z}$. Hence the natural morphism of graded Lie algebras

3.4.1. $$gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2l}]; l) \to \bigoplus_{i=1}^{\infty} (G_i(V, \vec{01})/G_{i+1}(V, \vec{01})) \otimes \mathbb{Q}$$

is surjective (see [10], Proposition 19.1,ii)). Moreover the natural morphism

3.4.2. $$\bigoplus_{i=1}^{\infty}(G_i(V,\overrightarrow{01})/G_{i+1}(V,\overrightarrow{01}))\otimes\mathbb{Q}\to(Lie(X,Y_0,Y_1),\{\ \})$$

is injective (see [10], Proposition 19.2,i)). The morphism 3.2 is the composition of morphisms 3.4.1 and 3.4.2. It follows from Proposition 1.2 iii) that the morphism 3.2 induces a morphism 3.3. Hence the morphism 3.3 induces a surjective morphism of graded Lie algebras

3.4.3. $$gr_W\,Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l\to\bigoplus_{i=1}^{\infty}(G_i(V,\overrightarrow{01})/G_{i+1}(V,\overrightarrow{01}))\otimes\mathbb{Q}$$

Both graded Lie algebras are free, freely generated by elements dual to $\kappa(2)$ and $l_{2n+1}(-1)$ for $n>0$. Hence the morphism 3.4.3 is an isomorphism. This implies that the morphism

$$(gr_W\,Lie\varphi_{\overrightarrow{01}})_l : gr_W\,Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l\to(Lie(X,Y_0,Y_1),\{\ \})$$

is injective. □

The immediate consequence of Proposition 3.4 is the following corollary.

Corollary 3.5. All coefficients on $gr_W\,Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l$ are geometrical, more precisely

$$(gr_W\,Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l)^{\diamond}=GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,-1,\infty\},\overrightarrow{01}).$$

We recall that the morphism of graded Lie algebras

$$gr_W\,Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l\to gr_W\,Lie\mathcal{U}(\mathbb{Z})_l$$

induced by the inclusion of rings $\mathbb{Z}\hookrightarrow\mathbb{Z}[\frac{1}{2}]$ is surjective and its kernel is by the definition the Lie ideal $I(\mathbb{Z}[\frac{1}{2}]:\mathbb{Z})$ (see Proposition 1.6). Hence Corollary 3.5 implies the following result.

Corollary 3.6. The vector space of coefficients on $gr_W\,Lie\mathcal{U}(\mathbb{Z})_l$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}}\setminus 0,1,-1,\infty\},\overrightarrow{01})$ consisting of all coefficients unramified everywhere.

Remark 3.6.1. The corresponding statement in Hodge-De Rham realization says that all periods of mixed Tate motives over Spec$\mathbb{Z}$ are unramified everywhere $\mathbb{Q}$-linear combinations of iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,-1,\infty\}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ in one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$.

Now we shall look more carefully at geometric coefficients to see which are unramified everywhere.

The Lie algebra $gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l$ is free, freely generated by one generator z_i in each odd degree. The Lie ideal $I(\mathbb{Z}[\frac{1}{2}]:\mathbb{Z})$ is generated by the generator in degree 1. This generator z_1 can be chosen to be dual to the Kummer character $\kappa(2)$, i.e. $\kappa(2)(z_1)=1$.

Let us choose a Hall base $\mathcal{B}$ of a free Lie algebra $Lie(X,Y_0,Y_1)$. Then the geometric coefficients, elements of the $\mathbb{Q}_l$-vector space $GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,-1,\infty\},\overrightarrow{01})$ are of the form $\{\overrightarrow{10},\overrightarrow{01}\}_{e^*}$ and $\{\overrightarrow{10},\overrightarrow{01}\}_\psi$, where $\psi=\sum_{i=1}^k n_i e_i^*$ and $e,e_i\in\mathcal{B}$.

Proposition 3.7. Let $e\in\mathcal{B}$ be a Lie bracket in X and Y_0 only. Then the coefficients $\{\overrightarrow{10},\overrightarrow{01}\}_{e^*}$ is unramified everywhere.

Proof. Let $j:\mathbb{P}^1\setminus\{0,1,-1,\infty\}\hookrightarrow\mathbb{P}^1\setminus\{0,1,\infty\}$ be the inclusion. Then j induces a morphism of fundamental groups based at $\overrightarrow{01}$. After the standard embeddings of fundamental groups into non commutative formal power series we get a morphism of $\mathbb{Q}_l$-algebras

$$j_*:\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}\to\mathbb{Q}_l\{\{X,Y\}\}$$

such that $j_*(X)=(X),\ j_*(Y_0)=Y,\ j_*(Y_1)=0$.

Then we have $\{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y_0)^*}=\{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y)^*\circ j_*}=\{j(\overrightarrow{10}),j(\overrightarrow{01})\}_{e(X,Y)^*}=\{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y)^*}$ (see [8] (10.0.6)). The pair $(\mathbb{P}^1\setminus\{0,1,\infty\},\overrightarrow{01})$ is unramified everywhere, hence the coefficient $\{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y_0)^*}$ belonging to $GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1\setminus\{0,1,-1,\infty\},\overrightarrow{01})$ is unramified everywhere. □

There are however coefficients in the $\mathbb{Q}_l$-vector space $GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1\setminus\{0,1,-1,\infty\},\overrightarrow{01})$ which contain Y_1 and which also are unramified everywhere. These coefficients are of course the most interesting in view of Corollary 3.6 as we perhaps still do not know if the inclusion

$$GeomCoeff^l_{\mathbb{Z}}(\mathbb{P}^1\setminus\{0,1,\infty\},\overrightarrow{01})\subset(gr_W Lie\mathcal{U}(\mathbb{Z})_l)^\diamond$$

is an equality. For example we have the following result.

Proposition 3.8. We have

$$\{\overrightarrow{10},\overrightarrow{01}\}_{[Y_1,X^{(n-1)}]^*}=\frac{1-2^{n-1}}{2^{n-1}}\cdot\{\overrightarrow{10},\overrightarrow{01}\}_{[Y_0,X^{(n-1)}]^*}.$$

Proof. It follows immediately from the definition of coefficients $\{\overrightarrow{10},\overrightarrow{01}\}_{e^*}$ and the definition of l-adic polylogarithms (see [8], Definition 11.0.1) that $\{\overrightarrow{10},\overrightarrow{01}\}_{[Y_0,X^{(n-1)}]^*}=l_n(1)$.

It follows from [9], Lemma 15.3.1 that $\{\overrightarrow{10},\overrightarrow{01}\}_{[Y_1,X^{(n-1)}]^*} = l_n(-1)$. The proposition now follows from the distribution relation $2^{n-1}(l_n(-1)+l_n(1)) = l_n(1)$ (see [8] Corollary 11.2.3). □

Below we shall give an inductive procedure to decide which coefficients are unramified everywhere. Let us denote for simplicity

$$\mathcal{L}_i := \mathcal{L}(\mathbb{Z}[\frac{1}{2}];l)_i \quad \text{and} \quad \mathcal{L}_{>1} := \bigoplus_{i=2}^{\infty} \mathcal{L}_i.$$

Lemma 3.9. We have

i) $\mathcal{L}_i = \mathbb{Q}_l$ for i odd and $\mathcal{L}_i = 0$ for i even;
ii) $\mathcal{L}_1$ is generated by the Kummer character $\kappa(2)$;
iii) $\mathcal{L}_{2k+1}$ is generated by $l_{2k+1}(-1)$ for $k > 0$.

Proof. The point i) follows from the isomorphisms $\mathcal{L}_1 \approx \mathbb{Z}[\frac{1}{2}]^* \otimes \mathbb{Q}_l$ in degree 1 and $\mathcal{L}_i \approx H^1(G_{\mathbb{Q}};\mathbb{Q}_l(i))$ for $i > 1$. Hence $\mathcal{L}_1$ is generated by the Kummer character $\kappa(2)$. The cohomology group $H^1(G_{\mathbb{Q}};\mathbb{Q}_l(2k+1))$ is generated by a Soulé class, which is a rational multiple of $l_{2k+1}(-1)$. □

If $e \in \mathcal{B}$ then $\deg_{Y_i} e$ denotes degree of e with respect to Y_i. We define

$$\deg_Y e \; := \; \deg_{Y_0} e \; + \; \deg_{Y_1} e.$$

Lemma 3.10. Let $\varphi \in GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0,1,-1,\infty\},\overrightarrow{01})$ be homogenous of degree k.

i) If $k = 1$ then $d\varphi = 0$ and φ is a $\mathbb{Q}_l$-multiple of $\kappa(2)$. Hence if $\varphi \neq 0$ then φ ramifies at (2);
ii) If $k > 1$ and $d\varphi = 0$ the φ is unramified everywhere;
iii) If $k > 1$ and $\varphi = \sum_{i=1}^m a_i e_i^*$, where $e_i \in \mathcal{B}$ and $\deg_Y e_i^* = 1$ for each i then $d\varphi = 0$ and φ is unramified everywhere.

Proof. In degree 1 there are the following geometric coefficients $\{\overrightarrow{10},\overrightarrow{01}\}_X = 0$, $\{\overrightarrow{10},\overrightarrow{01}\}_{Y_0} = 0$ and $\{\overrightarrow{10},\overrightarrow{01}\}_{Y_1} = \kappa(2)$ – the Kummer character of 2, which ramifies at (2).

If $\deg\varphi = k > 1$ and $d\varphi = 0$ then φ is a $\mathbb{Q}_l$-multiple of $l_k(-1)$ by Lemma 3.9 iii). Hence φ is unramified everywhere by Propositions 3.8 and 3.7.

If $\deg_Y e = 1$ then $e = [Y_0, X^{(k-1)}]$ or $e = [Y_1, X^{(k-1)}]$. In both cases it is clear that $d(e^*) = 0$. □

Lemma 3.11. Let $\varphi \in GeomCoeff^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0,1,-1,\infty\},\overrightarrow{01})$ be homogenous of degree $k > 1$.

i) If $d^{(2)}\varphi = 0$ then $d\varphi \in \mathcal{L} \otimes \mathcal{L}$;

ii) If $d^{(2)}\varphi = 0$ then φ is unramified everywhere if and only if $d\varphi \in \mathcal{L}_{>1}\otimes\mathcal{L}_{>1}$;

iii) If $\varphi = \sum_{i=1}^m n_i\{\vec{10},\vec{01}\}_{e_i^*}$, where $e_i \in \mathcal{B}$ and $\deg_Y e_i \leq 2$ for each i then $d^{(2)}(\varphi) = 0$.

Proof. We can write $d\varphi = \sum_{i\in I}\alpha_i\otimes\beta_i$, where α_i and β_i belong to $(gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l)^\diamond$ for $i \in I$ and β_i are linearly independent. Hence $d^{(2)}(\varphi) = \sum_{i\in I} d(\alpha_i)\otimes\beta_i = 0$ implies that $d(\alpha_i) = 0$ for $i \in I$. Hence it follows that $\alpha_i \in \mathcal{L}$. Therefore we can write $d\varphi = \sum_{n=1}^\infty a_n\otimes\varphi_n$, where $a_n \in \mathcal{L}_n$ because $\dim\mathcal{L}_n \leq 1$ by Lemma 3.9. It follows from the equality $d^{(2)}\varphi = 0$ that $(Id\otimes d)(d\varphi) = 0$. Hence $\varphi_n \in \mathcal{L}$ for each n. This implies that $d\varphi \in \mathcal{L}\otimes\mathcal{L}$.

We recall that the Lie algebra $gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{2}])_l$ is free, freely generated by elements z_k of degree k for k odd.

If φ is unramified then clearly $d\varphi \in \mathcal{L}_{>1}\otimes\mathcal{L}_{>1}$. Let us suppose that $\deg\varphi = k > 1$ and $d\varphi \in \mathcal{L}_{>1}\otimes\mathcal{L}_{>1}$. Then $\varphi(z_1) = 0$ because $\deg\varphi > 1$ and $\varphi([z_1, z_i]) = 0$ because $d\varphi \in \mathcal{L}_{>1}\otimes\mathcal{L}_{>1}$.

The condition $d^{(2)}\varphi = 0$ implies that φ vanishes on $\Gamma^3 gr_W Lie\mathcal{U}(\mathbb{Z}([\frac{1}{2}])_l$. Hence φ vanishes on the Lie ideal $I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})$. Therefore φ is unramified everywhere.

If $e \in \mathcal{B}$ and $\deg_Y e = 2$ then $d(\{\vec{10},\vec{01}\}_{e^*}) \in \mathcal{L}\otimes\mathcal{L}$, hence $d^{(2)}(\{\vec{10},\vec{01}\}_{e^*}) = 0$. □

4 $\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\}\cup\mu_3)$ and periods of mixed Tate motives over $\mathrm{Spec}\mathbb{Z}[\frac{1}{3}]$ and $\mathrm{Spec}\mathbb{Z}[\mu_3]$

Let $U := \mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\}\cup\mu_3)$. In [9] we have also studied the Galois representation

$$\varphi_{U,\vec{01}} : G_{\mathbb{Q}(\mu_3)} \longrightarrow Aut\big(\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\}\cup\mu_3); \vec{01})\big).$$

The pair $(U,\vec{01})$ has good reduction outside the prime ideal $(1-\xi_3)$ of $\mathcal{O}_{\mathbb{Q}(\mu_3)}$, where ξ_3 is a primitive 3rd root of 1. Observe that we have an equality of ideals $(1-\xi_3)^2 = (3)$. Hence we get a morphism of graded Lie algebras

4.0. $$gr_W Lie\varphi_{U,\vec{01}} : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_3][\frac{1}{3l}];l) \longrightarrow (Lie(X,Y_0,Y_1,Y_2),\{\ \}).$$

It follows from Proposition 1.2, iii) that the morphism 4.0 induces a morphism

4.1. $$(gr_W Lie\varphi_{U,\vec{01}})_l : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_3][\frac{1}{3}])_l \longrightarrow (Lie(X,Y_0,Y_1,Y_2),\{\ \}).$$

Proposition 4.2. The morphism of graded Lie algebras

$$(gr_W Lie\varphi_{U,\vec{01}})_l : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_3][\frac{1}{3}])_l \longrightarrow (Lie(X,Y_0,Y_1,Y_2),\{\ \}).$$

deduced from the action of $G_{\mathbb{Q}(\mu_3)}$ on $\pi_1(U_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ is injective.

Proof. The proposition follows from [9], Theorem 15.4.7. □

Corollary 4.3. All coefficients on $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_3][\frac{1}{3}])_l$ are geometrical. More precisely we have

$$(gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_3][\frac{1}{3}])_l)^\diamond = GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}).$$

Proof. The result follows immediately from Proposition 4.2. □

Corollary 4.4. We have:

i) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_3])_l)^\diamond$ is equal to the vector subspace of these elements of $GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$, which are unramified everywhere;

ii) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{3}])_l)^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$ consisting of coefficients which are defined over $\mathbb{Q}$;

iii) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z})_l)^\diamond$ is equal to the vector subspace of these elements of $GeomCoeff^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1 \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$, which are defined over $\mathbb{Q}$ and unramified everywhere.

Proof. The corollary follows from Corollary 4.3 and from Proposition 1.12 and Proposition 1.6. □

5 $\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4)$ and $\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8)$ and periods of mixed Tate motives over Spec$\mathbb{Z}[i]$, Spec$\mathbb{Z}[\mu_8]$, Spec$\mathbb{Z}[\sqrt{2}][\frac{1}{2}]$, Spec$\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]$, Spec$\mathbb{Z}[\sqrt{2}]$ and Spec$\mathbb{Z}[\sqrt{-2}]$

The pair $(\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4), \overrightarrow{01})$ (resp. $(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \overrightarrow{01})$) has good reduction outside the prime ideal $(1-i)$ of $\mathbb{Z}[\mu_4]$ (resp. $(1-e^{\frac{2\pi i}{8}})$ of $\mathbb{Z}[\mu_8]$) lying over (2). Hence it follows from Proposition 1.2 iii) and from [9], Corollary 15.6.4 and Proposition 15.6.5 that morphisms of graded Lie algebras

$$(gr_W Lie\varphi_{\overrightarrow{01}})_l : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_4][\frac{1}{2}])_l \to (Lie(X, Y_0, Y_1, Y_2, Y_3), \{\ \})$$

and

$$(gr_W Lie\varphi_{\overrightarrow{01}})_l : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_8][\frac{1}{2}])_l \to (Lie(X, Y_0, Y_1, \dots, Y_8), \{\ \})$$

deduced from the action of $G_{\mathbb{Q}(\mu_4)}$ (resp. $G_{\mathbb{Q}(\mu_8)}$) on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_4); \vec{01})$ (resp. $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_8); \vec{01})$) are injective. Hence we get

Proposition 5.1. All coefficients on $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_4][\frac{1}{2}])_l$ and on $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_8][\frac{1}{2}])_l$ are geometrical, more precisely

$$(gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_4][\frac{1}{2}])_l)^\diamond = GeomCoeff^l_{\mathbb{Z}[\mu_4][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4), \vec{01})$$

and

$$(gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_8][\frac{1}{2}])_l)^\diamond = GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \vec{01}).$$

Corollary 5.2.

i) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_4])_l)^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_4][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4), \vec{01})$ consisting of the coefficients which are unramified everywhere;

ii) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_8])_l)^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \vec{01})$ consisting of the coefficients which are unramified everywhere;

iii) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\sqrt{2}][\frac{1}{2}])_l)^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \vec{01})$ consisting of coefficients which are defined over $\mathbb{Q}(\sqrt{2})$;

iv) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\sqrt{2}])_l)^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \vec{01})$ consisting of coefficients which are unramified everywhere and defined over $\mathbb{Q}(\sqrt{2})$;

v) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}])_l)^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \vec{01})$ consisting of coefficients which are defined over $\mathbb{Q}(\sqrt{-2})$;

vi) The vector space $(gr_W Lie\mathcal{U}(\mathbb{Z}[\sqrt{-2}])_l)^\diamond$ is equal to the vector subspace of $GeomCoeff^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \vec{01})$ consisting of coefficients which are unramified everywhere and defined over $\mathbb{Q}(\sqrt{-2})$.

6 Periods of mixed Tate motives, Hodge–De Rham side

We shall give here a sketch of a proof of the result annouced at the beginning of the paper.

Theorem 6.1. The $\mathbb{Q}$-algebra of periods of mixed Tate motives over Spec$\mathbb{Z}$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,-1,\infty\}$ of sequences of one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are unramified everywhere.

Sketch of a proof. We shall assume formalism from [1]. In the group of complex points of the affine group $U_{DR}(\mathbb{Z}[\frac{1}{2}])$ there is an element $\gamma_{\mathbb{Z}[\frac{1}{2}]}\in U_{DR}(\mathbb{Z}[\frac{1}{2}])(\mathbb{C})$ which gives all rational lattices in the Hodge - De Rham realization of mixed Tate motives over Spec$\mathbb{Z}[\frac{1}{2}]$. The action of $U_{DR}(\mathbb{Z}[\frac{1}{2}])$ on the mixed Tate motive associated to $\pi_1(\mathbb{P}^1(\mathbb{C})\setminus\{0,1,-1,\infty\};\overrightarrow{01})$ is free. This is an analog of Proposision 3.4. (See talk of P. Deligne in Schloss Ringberg ([2]) and also [12], where cases of $\pi_1(\mathbb{P}^1(\mathbb{C})\setminus(\{0,\infty\}\cup\mu_3);\overrightarrow{01})$ and $\pi_1(\mathbb{P}^1(\mathbb{C})\setminus(\{0,\infty\}\cup\mu_4);\overrightarrow{01})$ were considered.) Hence the coordinates of $\gamma_{\mathbb{Z}[\frac{1}{2}]}$are given by iterated integrals on $\mathbb{P}^1\setminus\{0,1,-1,\infty\}$ of sequences of one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$.

The natural embedding of the category of mixed Tate motives over Spec$\mathbb{Z}$ into the category of mixed Tate motives over Spec$\mathbb{Z}[\frac{1}{2}]$ induces a surjective morphisme

$$U_{DR}(\mathbb{Z}[\frac{1}{2}])\longrightarrow U_{DR}(\mathbb{Z}).$$

Let $\bar\gamma$ be the image of $\gamma_{\mathbb{Z}[\frac{1}{2}]}$ in $U_{DR}(\mathbb{Z})(\mathbb{C})$. Then $\bar\gamma$ is an element which gives all rational lattices in the Hodge - De Rham realization of mixed Tate motives over Spec$\mathbb{Z}$. Passing to Lie algebras we get a morphism of Lie algebras

$$LieU_{DR}(\mathbb{Z}[\frac{1}{2}])\longrightarrow LieU_{DR}(\mathbb{Z}),$$

which maps $log\gamma_{\mathbb{Z}[\frac{1}{2}]}$ into $log\bar\gamma$. Let φ be a linear form on $LieU_{DR}(\mathbb{Z})$. Then φ evaluated on $\bar\gamma$ gives a period of a mixed Tate motive over Spec$\mathbb{Z}$. The linear form φ view as a linear form on $LieU_{DR}(\mathbb{Z}[\frac{1}{2}])$ vanishes on the ideal $I(\mathbb{Z}[\frac{1}{2}]:\mathbb{Z})$. Hence $\varphi(\bar\gamma)$ is unramified everywhere and $\varphi(\bar\gamma)$ is a sum of iterated integrals on $\mathbb{P}^1\setminus\{0,1,-1,\infty\}$ of sequences of one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. □

7 An example of a missing coefficient

Let p be a prime number. In [11] we have study the problem of expressing coefficients of $(gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{p}])_l)^\diamond$ by geometrical coefficients. Using the action of $G_{\mathbb{Q}(\mu_p)}$ on the torsor of paths $\pi(\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\};\xi_p,\overrightarrow{01})$ we have constructed all coefficients on $gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{p}])_l$. After publishing the paper [11] we realize that it is far from obvous that the coefficients we constructed are geometrical and moreover to show that they are geometrical we should rather look at the action of $G_{\mathbb{Q}(\mu_p)}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus(\{0,\infty\}\cup\mu_p);\overrightarrow{01})$.

We hope studing the action of $G_{\mathbb{Q}(\mu_p)}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_p); \overrightarrow{01})$ to show that results of [11] concerning Spec$\mathbb{Z}[\frac{1}{p}]$ are true. More precisely we hope to show that the kernel of the morphism

$$gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_p][\frac{1}{p}])_l \longrightarrow (Lie(X, Y_0, \dots, Y_{p-1}), \{\ \})$$

deduced from the action of $G_{\mathbb{Q}(\mu_p)}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_p); \overrightarrow{01})$ is contained in the Lie ideal $I(\mathbb{Z}[\mu_p][\frac{1}{p}] : \mathbb{Z}[\frac{1}{p}])$. Then it implies that all coefficients of $(gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_p][\frac{1}{p}])_l)^{\diamond}$ are geometrical. In the special cases considered in the present paper it is true because the kernel for $p = 2$, $p = 3$ and also for $p = 4$ and $p = 8$ is zero.

We finish our paper with an example showing that one can deal with a single coefficient. We shall use notations and results from our papers [9] and [10].

Let p be an odd prime. The pair $(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p), \overrightarrow{01})$ has good reduction outside (p). Hence the action of $G_{\mathbb{Q}(\mu_p)}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_p); \overrightarrow{01})$ leads to a Lie algebra homomorphism

$$gr_W Lie\varphi_{\overrightarrow{01}} : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_p][\frac{1}{p}])_l \longrightarrow Der^*_{\mathbb{Z}/p}(Lie(X, Y_0, \dots, Y_{p-1})).$$

The following result generalize our partial results for $p = 5$ (see [10], Proposition 20.5) and for $p = 7$ (see [3], Theorem 4.1).

Proposition 7.1. Let p be an odd prime.

i) In the image of the morphism of graded Lie algebras

$$gr_W Lie\varphi_{\overrightarrow{01}} : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_p][\frac{1}{p}])_l \longrightarrow Der^*_{\mathbb{Z}/p}(Lie(X, Y_0, \dots, Y_{p-1})).$$

there are linearly independent over $\mathbb{Q}_l$ derivations τ_i for $1 \leq i \leq \frac{p-1}{2}$ such that

$$\tau_i(Y_0) = [Y_0, Y_i + Y_{p-i}].$$

ii) There are the following relations between commutators

$$\mathcal{R}_k : \quad [\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i] = 0 \quad \text{for} \quad 1 \leq k \leq \frac{p-1}{2}$$

and between relations

$$\sum_{i=k}^{\frac{p-1}{2}} \mathcal{R}_k = 0.$$

Proof. The equality $\xi_p^i(1-\xi_p^{p-i}) = -(1-\xi_p^i)$ implies that $l(1-\xi_p^{p-i}) = l((1-\xi_p^i)$ on $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_p][\frac{1}{p}])_l$. Elements $1-\xi_p^i$ for $1 \leq i \leq \frac{p-1}{2}$ are linearly independent in the $\mathbb{Z}$-module $\mathbb{Z}[\mu_p]^*$. The point i) of the proposition follows from [9], Lemma 15.3.2.

To show ii) we calculate $[\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i] = \{Y_k + Y_{p-k}, \sum_{i=1}^{\frac{p-1}{2}} (Y_i + Y_{p-i})\} = \{Y_k + Y_{p-k}, \sum_{i=0}^{p-1} Y_i\} = [Y_k, \sum_{i=0}^{p-1} Y_i] + \sum_{i=0}^{p-1} [Y_i, Y_{i+k}] - \sum_{i=0}^{p-1} [Y_k, Y_{k+i}] + [Y_{p-k}, \sum_{i=0}^{p-1} Y_i] + \sum_{i=0}^{p-1} [Y_i, Y_{i+p-k}] - \sum_{i=0}^{p-1} [Y_{p-k}, Y_{i+p-k}] = 0$.

The relation $[\sum_{k=1}^{\frac{p-1}{2}} \tau_k, \sum_{i=1}^{\frac{p-1}{2}} \tau_i] = 0$ holds in any Lie algebra, hence we have a relation $\sum_{k=1}^{\frac{p-1}{2}} \mathcal{R}_k = 0$ between the relations. □

In the Lie algebra $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_p][\frac{1}{p}])_l$ we have $\frac{p-1}{2}$ linearly independent generators $T_1, \ldots, T_{\frac{p-1}{2}}$ in degree 1, which generate a free Lie subalgebra. Hence the obvious question is how to construct geometric coefficients in degree 2 (periods of mixed Tate motives over Spec$\mathbb{Z}[\mu_p][\frac{1}{p}]$) which are dual to Lie brackets $[T_i, T_j]$ for $(i < j)$. It is clear from Proposition 7.1 that there is not enough coefficients in $GeomCoeff^l_{\mathbb{Z}[\mu_p][\frac{1}{p}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p), \overrightarrow{01}\big)$.

We consider only the simplest case $p = 5$. We start with the action of $G_{\mathbb{Q}(\mu_{10})} = G_{\mathbb{Q}(\mu_5)}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{10}), \overrightarrow{01})$. Observe that the pair $(\mathbb{P}^1_{\mathbb{Q}(\mu_{10})} \setminus (\{0,\infty\} \cup \mu_{10}), \overrightarrow{01})$ has good reduction outside divisors of (10). We have the following result.

Proposition 7.2. We have:

i) In degree 1 the image of the morphism

$$gr_W Lie\varphi_{\overrightarrow{01}} : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_5][\frac{1}{10}])_l \longrightarrow (Lie(X, Y_0, \ldots, Y_9), \{\ \})$$

is generated by $\sigma_1 := Y_1 + Y_9 + Y_2 + Y_8 - Y_3 - Y_7$, $\sigma_2 := -Y_1 - Y_9 + Y_4 + Y_6 + Y_3 + Y_7$ and $\eta := Y_5$.

ii) The Lie bracket $\{\sigma_1, \sigma_2\}$ is different from zero and the coefficient of $\{\sigma_1, \sigma_2\}$ at $[Y_1, Y_8]$ is 1.

iii) Let $f := [Y_1, Y_8]^*$ be a linear form on $Lie(X, Y_0, \ldots, Y_9)$ dual to $[Y_1, Y_8]$ with respect to standard Hall base of $Lie(X, Y_0, \ldots, Y_9)$. Let

$$\mathcal{F} := f \circ gr_W Lie\varphi_{\overrightarrow{01}}.$$

Then $\mathcal{F}$ vanishes on the Lie ideal $I(\mathbb{Z}[\mu_5][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$. Hence $\mathcal{F}$ defines a non trivial linear form of degree 2 on $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_5][\frac{1}{5}])_l$ non vanishing on $\Gamma^2 gr_W Lie\mathcal{U}(\mathbb{Z}[\frac{1}{5}])_l$.

Proof. We omit the proof of i) which is standard as in our other papers. We notice only that σ_1, σ_2 and η is a base dual to l-adic logarithms (i.e, Kummer characters) $l(1-\xi_5^1)$, $l(1-\xi_5^2)$ and $l(2)$. To show ii) we just calculate. It rests to show iii).

In the Lie algebra $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_5][\frac{1}{10}])_l$ (resp. $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_5][\frac{1}{5}])_l$) we choose in degree 1 generators S_1, S_2, N (resp. T_1, T_2) dual to $l(1-\xi_5^1)$, $l(1-\xi_5^2)$ and $l(2)$ (resp. $l(1-\xi_5^1)$ and $l(1-\xi_5^2)$). Then the natural morphism

$$\pi : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_5][\frac{1}{10}])_l \longrightarrow gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_5][\frac{1}{5}])_l$$

is given in degree 1 by $\pi(S_i) = T_i$ and $\pi(N) = 0$.

The Lie ideal $I(\mathbb{Z}[\mu_5][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$ in degree 2 has a base $[S_1, N]$, $[S_2, N]$. The Lie algebra morphism

$$gr_W Lie\varphi_{\overrightarrow{01}} : gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_5][\frac{1}{10}])_l \longrightarrow Lie(X, Y_0, \ldots, Y_9), \{\ \})$$

in degree 1 is given by $(gr_W Lie\varphi_{\overrightarrow{01}})(S_1) = \sigma_1$, $(gr_W Lie\varphi_{\overrightarrow{01}})(S_2) = \sigma_2$ and $(gr_W Lie\varphi_{\overrightarrow{01}})(N) = \eta$.

Observe that the linear form $[Y_1, Y_8]^*$ vanishes on $\{\sigma_1, \eta\}$ and on $\{\sigma_2, \eta\}$. Therefore $\mathcal{F}$ vanishes on $[S_1, N]$ and on $[S_2, N]$. Hence $\mathcal{F}$ vanishes on the Lie ideal $I(\mathbb{Z}[\mu_5][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$. Therefore $\mathcal{F}$ defines a linear form $\bar{\mathcal{F}}$ on $gr_W Lie\mathcal{U}(\mathbb{Z}[\mu_5][\frac{1}{5}])_l$ such that $\bar{\mathcal{F}}([S_1, S_2]) = 1$ because $[Y_1, Y_8]^*(\{\sigma_1, \sigma_2\}) = 1$. □

Remark 7.3. There are three linearly independent periods of mixed Tate motives over Spec$\mathbb{Z}[\mu_5][\frac{1}{5}]$ in degree 2, $Li_2(\xi_5^1)$, $Li_2(\xi_5^2)$ and the iterated integral $\int_0^1 \frac{dz}{z-\xi_{10}^1}, \frac{dz}{z-\xi_{10}^8}$. On the other hand one cannot get this third period as an iterated integral on $\mathbb{P}^1(\mathbb{C}) \setminus (\{0, \infty\} \cup \mu_5)$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ of the sequence of one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$, $\frac{dz}{z-\xi_5^k}$ for $k = 1, 2, 3, 4$.

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