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On the distribution of sequential Hölder norms of the Brownian motion*

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Abstract

The distributions of Hölder norms of Brownian motion and of Brownian bridge are limiting distributions (under the null hypothesis) of some statistics based on uniform increments of partial sums process allowing to detect some short "epidemic" changes in a sample. Unfortunately the exact distribution of these norms is not known. For practical reasons it is then convenient to use dyadic increment statistics whose limiting distribution is the one of sequential Hölder norms of the Brownian bridge. The aim of this paper is to study the practical computations of such distributions.

Résumé

Les lois de normes hölderiennes du mouvement et du pont browniens sont les lois limites (sous l'hypothèse nulle) de certaines statistiques de test basées sur les accroissements uniformes de processus de sommes partielles. Ces tests permettent la détection de ruptures « épidémiques » dans un échantillon. Malheureusement la loi exacte de ces normes n'est pas connue. Pour des raisons pratiques, il est commode d'utiliser à la place des statistiques basées sur les accroissements dyadiques et dont la loi limite est celle de normes hölderiennes séquentielles du pont brownien. Notre but est d'étudier le calcul effectif d'une telle loi.

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1 Introduction

The use of dyadic increments statistics together with Hölderian invariance principle are useful to detect short epidemics [16, 17]. To be more precise, let us just sketch the simplest case. Suppose that $X_1, \dots, X_n$ are independent random variables with means $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ respectively. We want to test the standard null hypothesis of a constant mean

(H_0): $\mathfrak{m}_1 = \dots = \mathfrak{m}_n$

against the so called epidemic alternative

(H_A): *there are integers* $1 < k^* < m^* < n$ *such that*
$\mathfrak{m}_1 = \mathfrak{m}_2 = \dots = \mathfrak{m}_{k^*} = \mathfrak{m}_{m^*+1} = \dots = \mathfrak{m}_n$,
$\mathfrak{m}_{k^*+1} = \dots = \mathfrak{m}_{m^*}$ *and* $\mathfrak{m}_{k^*} \neq \mathfrak{m}_{k^*+1}$.

The study of epidemic change models goes back to Levin and Kline [10] who proposed test statistics based on partial sums. Using a maximum likelihood approach, Yao [21] suggested some weighted versions of these statistics which may be viewed as some discrete Hölder norm of the partial sums process, see also Csörgő and Horváth [3] and the references therein.

In [16], we study a large class of statistics obtained by discretizing Hölder norms of the partial sums process. One important feature of Hölderian weighting is the detection of short epidemics. Roughly speaking, the use of Hölderian tests allows the detection of epidemics whose length $l^* := m^* - k^*$ is at least of the order of $\ln^\gamma n$ with $\gamma > 1$, while the same test statistics without Hölderian weight detects only epidemics such that $n^{-1/2} l^*$ goes to infinity. Among the test statistics suggested in [16], the statistics $\mathrm{DI}(n, \rho)$ built on the dyadic increments of partial sums are of special interest because their limiting distribution is explicitly computable.

Let us denote by D_j the set of dyadic numbers in $[0, 1]$ of level j, i.e.

$$\mathrm{D}_0 = \{0, 1\}, \qquad \mathrm{D}_j = \big\{(2l-1)2^{-j};\ 1 \le l \le 2^{j-1}\big\}, \quad j \ge 1.$$

So the countable set D of dyadic numbers of $[0, 1]$ may be written as

$$\mathrm{D} = \bigcup_{j=0}^{\infty} \mathrm{D}_j.$$

We use the enumeration $n \mapsto r_n$ of the elements of D provided by the lexicographic order on the couples (j, r) where j is the level of r. So $r_0 = 0$, $r_1 = 1$, $r_2 = 1/2$, $r_3 = 1/4$, $r_4 = 3/4$, $r_5 = 1/8, \dots$ Hence a series like $\sum_{r \in \mathrm{D}} f(r)$ should be understood as $\sum_{n=0}^{\infty} f(r_n)$. For notational convenience, we put

$$\mathrm{D}^* := \mathrm{D} \setminus \{0\}.$$

Write for $r \in \mathrm{D}_j$, $j \ge 0$,

$$r^- := r - 2^{-j}, \quad r^+ := r + 2^{-j}.$$

For a function $x : [0,1] \to \mathbb{R}$, we shall denote

$$\lambda_r(x) := \begin{cases} x(r) - \frac{1}{2}(x(r^+) + x(r^-)), & \text{if } r \in \mathrm{D}_j,\ j \geq 1, \\ x(r) & \text{if } r \in \mathrm{D}_0. \end{cases} \tag{1}$$

Consider partial sums

$$S(0) = 0, \quad S(u) = \sum_{k \leq u} X_k, \quad 0 < u < \infty.$$

We define also

$$S_n(a,b) := S(nb) - S(na) = \sum_{na < k \leq nb} X_k, \quad 0 \leq a < b \leq 1$$

and

$$S_n(t) = S_n(0,t) = \sum_{k \leq nt} X_k, \quad 0 \leq t \leq 1.$$

Dyadic increments statistics $\mathrm{DI}(n,\rho)$ depend on a weight function $\rho : [0,1] \to \mathbb{R}$, of the form $\rho(h) = h^\alpha L(1/h)$ with L slowly varying, and are defined by

$$\begin{aligned} \mathrm{DI}(n,\rho) \quad &:= \max_{1 \leq j \leq \log n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \|\lambda_r(S_n)\| \\ &= \frac{1}{2} \max_{1 \leq j \log n} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} \Big\| \sum_{nr^- < k \leq nr} X_k - \sum_{nr < k \leq nr^+} X_k \Big\|. \end{aligned} \tag{2}$$

In this paper, we write "log" for the logarithm with basis 2 ($\log(2^j) = j$) and "ln" for the natural logarithm ($\ln(\mathrm{e}^t) = t$). The properties to be satisfied by the weight fonction ρ will be precised below. Let us just mention for the moment that the function $\rho(h) = h^\alpha$, $0 < \alpha < 1/2$ and $\rho(h) = \rho(h, 1/2, \beta) = h^{1/2} \ln^\beta(c/h)$ with $\beta > 1/2$ are the main practical examples we have in mind.

To obtain limiting distribution for $\mathrm{DI}(n,\rho)$, let us introduce a stronger null hypothesis, namely

> (H_0'): *$X_1, \dots, X_n$ are independent identically distributed random variables with mean denoted μ_0.*

Theorem 1 (Th. 3 in [16])**.** *Under (H_0'), assume that $\rho \in \mathcal{R}$ and for every $A > 0$,*

$$\lim_{t \to \infty} t\, \mathbf{P}(|X_1| > A\theta(t)) = 0, \quad \text{where } \theta(t) := t^{1/2}\rho(1/t). \tag{3}$$

Then

$$\sigma^{-1} n^{-1/2} \mathrm{DI}(n,\rho) \xrightarrow[n \to \infty]{\mathcal{D}} \mathrm{DI}(\rho), \tag{4}$$

where $\sigma^2 = \mathrm{Var}(X_1)$ and $\mathrm{DI}(\rho)$ is the sequential ρ-Hölder norm of the Brownian bridge B given by $\|B\|_\rho^{\mathrm{seq}} := \sup_{j \geq 0} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} |\lambda_r(B)|$.

When $\alpha < 1/2$, it is enough to take $A = 1$ in (3). In the special case $\rho(h) = h^\alpha$, (3) may be recast as

$$\mathbf{P}(|X_1| > t) = o(t^{-p}), \quad \text{with} \quad p := 1/(1/2 - \alpha).$$

In the other special case $\rho(h) = \rho(h, 1/2, \beta) = h^{1/2} \log^\beta(c/h)$, where $\beta > 1/2$, it is not possible to drop the constant A in Condition (3) which is easily seen to be equivalent to

$$\mathbf{E} \exp\{\lambda |X_1|^{1/\beta}\} < \infty \quad \text{for each} \quad \lambda > 0. \tag{5}$$

Our aim is to provide practical computations for the distribution function of $\mathrm{DI}(\rho)$. The paper is organized as follows. Section 2 presents briefly some analytical background on Hölder spaces, sequential norms, Haar functions, triangular Faber Schauder functions. This is used in Section 3 to generalize the classical Lévy-Kampé de Fériet expansion of Brownian motion in a series of triangular function to the Hölderian setting. In Section 4 this expansion leads to a representation of the distribution function of $\mathrm{DI}(\rho)$ as an infinite product of distributions functions. Section 5 discuss the rate of convergence of this product. Finally tables are given in Section 6 for the two above mentionned examples of weights ρ .

2 Haar and Faber-Schauder functions

A simple way to parametrize the collections of Haar and Faber-Schauder functions is to use directly the dyadic numbers. For $r \in \mathrm{D}_j$, $j \geq 1$, the $L^2[0,1]$ normalized Haar function H_r, is defined on $[0,1]$ by

$$H_r(t) = \begin{cases} +(r^+ - r^-)^{-1/2} = +2^{(j-1)/2} & \text{if } t \in (r^-, r]; \\ -(r^+ - r^-)^{-1/2} = -2^{(j-1)/2} & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

In the special case $j = 0$, as the support $[r^-, r^+]$ would extend beyond $[0,1]$ we have to modify the above formula to keep the $L^2[0,1]$ normalization. This leads us to define

$$H_0(t) := -\mathbf{1}_{[0,1]}(t), \quad H_1(t) := \mathbf{1}_{[0,1]}(t).$$

The sequence $\{H_r; r \in \mathrm{D}, r \neq 0\}$ is an orthonormal basis of the Hilbert space $L^2[0,1]$.

For $r \in \mathrm{D}_j$, $j \geq 1$, the triangular Faber-Schauder functions Λ_r are continuous, piecewise affine with support $[r^-, r^+]$ and taking the value 1 at r:

$$\Lambda_r(t) = \begin{cases} (t - r^-)/(r - r^-) = 2^j(t - r^-) & \text{if } t \in (r^-, r]; \\ (r^+ - t)/(r^+ - r) = 2^j(r^+ - t) & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

In the special case $j = 0$, we just take the restriction to $[0,1]$ in the above formula, so

$$\Lambda_0(t) = 1 - t, \quad \Lambda_1(t) = t, \quad t \in [0,1].$$

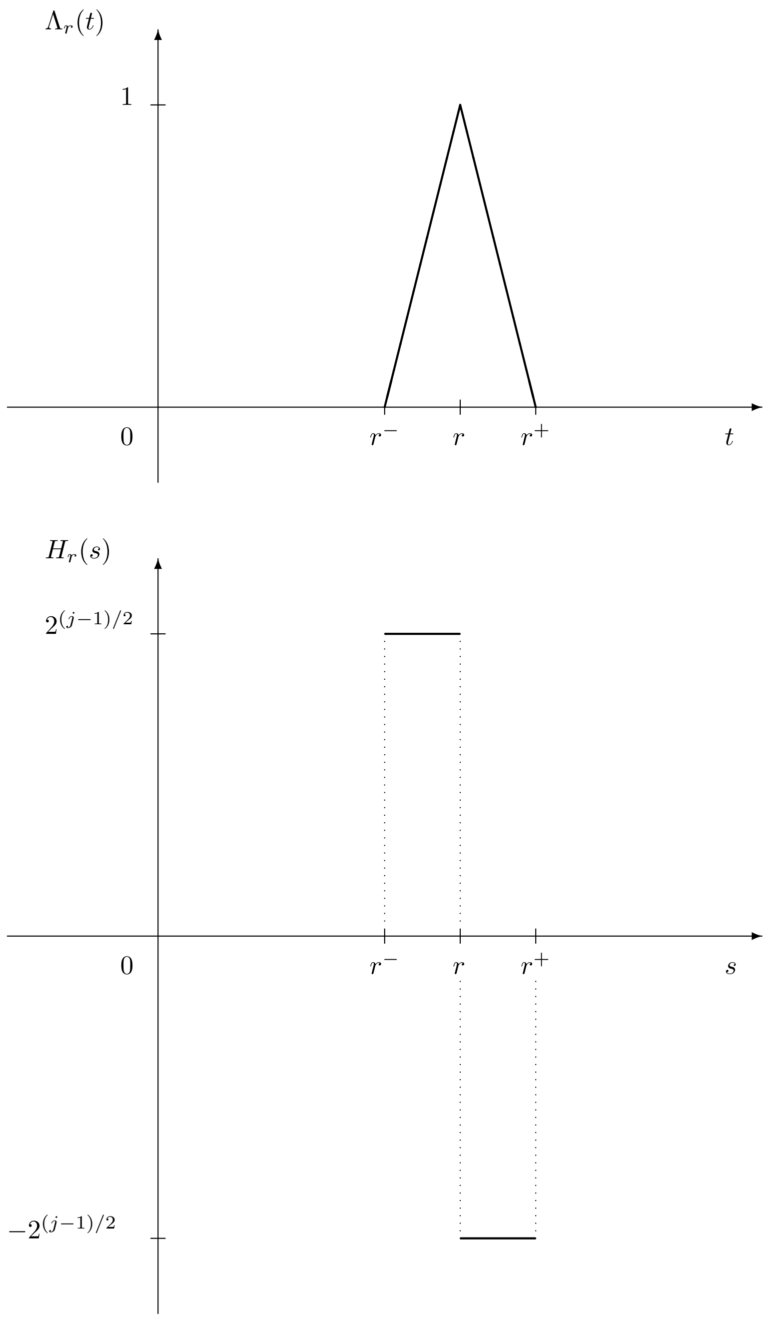

Figure 1: The Haar function H_r and the Faber-Schauder function Λ_r

The Λ_r's have the $C[0,1]$ normalization: $\|\Lambda_r\|_\infty = 1$. They are linked to the H_r's in the general case $r \in \mathrm{D}_j$, $j \geq 1$ by

$$\Lambda_r(t) = 2(r^+ - r^-)^{-1/2} \int_0^t H_r(s)\,\mathrm{d}s = 2^{(j+1)/2} \int_0^t H_r(s)\,\mathrm{d}s \tag{6}$$

and in the special case $j = 0$ by

$$\Lambda_0(t) = 1 + \int_0^t H_0(s)\,\mathrm{d}s, \quad \Lambda_1(t) = \int_0^t H_1(s)\,\mathrm{d}s. \tag{7}$$

The sequence $\{\Lambda_r; r \in \mathrm{D}\}$ is a Schauder basis of $C[0,1]$. Each $x \in C[0,1]$ has a unique expansion

$$x = \sum_{r\in\mathrm{D}} \lambda_r(x)\Lambda_r = \sum_{j=0}^{\infty} \sum_{r\in\mathrm{D}_j} \lambda_r(x)\Lambda_r,$$

with uniform convergence on $[0,1]$. The Schauder scalar coefficients $\lambda_r(x)$ are defined by (1).

The sequence $\{\Lambda_r; r \in \mathrm{D}\}$ is also a Schauder basis in a large class of Hölder spaces we are describing now. Let ρ be a real valued non decreasing function on $[0,1]$, null and right continuous at 0. Put

$$w_\rho(x,\delta) := \sup_{\substack{s,t\in[0,1],\\ 0<t-s<\delta}} \frac{|x(t)-x(s)|}{\rho(t-s)}.$$

We associate to ρ the separable Hölder space

$$\mathrm{H}^o_\rho[0,1] := \{x \in \mathrm{C}[0,1];\ \lim_{\delta\to 0} w_\rho(x,\delta) = 0\},$$

equiped with the norm

$$\|x\|_\rho := |x(0)| + w_\rho(x,1).$$

With the aim to use a sequential norm equivalent to $\|x\|_\rho$, we require, following Ciesielski (see e.g. [18, p.67]), that the modulus of smoothness ρ satisfies the conditions:

$$\rho(0) = 0,\ \rho(h) > 0,\ 0 < h \leq 1; \tag{8}$$

$$\rho \text{ is non decreasing on } [0,1]; \tag{9}$$

$$\rho(2h) \leq c_1\rho(h), \quad 0 \leq h \leq 1/2; \tag{10}$$

$$\int_0^h \frac{\rho(u)}{u}\,\mathrm{d}u \leq c_2\rho(h), \quad 0 < h \leq 1; \tag{11}$$

$$h\int_h^1 \frac{\rho(u)}{u^2}\,\mathrm{d}u \leq c_3\rho(h), \quad 0 < h \leq 1; \tag{12}$$

where c_1, c_2 and c_3 are positive constants. Let us observe in passing, that (8), (9) and (11) together imply the right continuity of ρ at 0. The class of functions ρ satisfying these requirements is rich enough according to the following.

Proposition 2. *For any* $0 < \alpha < 1$*, consider the function*

$$\rho(h) = h^{\alpha} L(1/h)$$

where L *is normalized slowly varying at infinity, continuous and positive on* $[1, \infty)$*. Then* ρ *fulfills conditions (8) to (12) up to a change of scale.*

The proof can be be found in [15, Prop.2].

Proposition 3. $\{\Lambda_r; r \in \mathrm{D}\}$ *is a Schauder basis of the space* $\mathrm{H}^o_\rho[0,1]$ *and we have the equivalence of norms*

$$\|x\|_\rho \sim \|x\|_\rho^{\mathrm{seq}} := \sup_{j \geq 0} \frac{1}{\rho(2^{-j})} \max_{r \in \mathrm{D}_j} |\lambda_r(x)|.$$

For a proof see [11] and [18].

3 Expansion of the Brownian motion

The expansion of the standard Brownian motion as a series of triangular functions converging in $C[0,1]$ is classical and goes back to Lévy and Kampé de Fériet. It turns out that the same series converges also in the stronger topology of $\mathrm{H}^o_\rho[0,1]$ for any ρ compatible with the smoothness of the Brownian motion. This provides a convenient expression of the ρ sequential norms of the Brownian motion and the Brownian bridge.

Theorem 4. *Assume that* ρ *fulfills Conditions (8) to (12) and that*

$$\sqrt{h|\ln h|} = o\big(\rho(h)\big), \quad h \to 0. \tag{13}$$

Let $\{X_r; r \in \mathrm{D}^*\}$ *be a sequence of independent* $\mathfrak{N}(0,1)$ *random variables. Then the random series of functions*

$$W := X_1\Lambda_1 + \sum_{j=1}^{\infty} \sum_{r \in \mathrm{D}_j} 2^{-(j+1)/2} X_r \Lambda_r, \tag{14}$$

converges a.s. in the space $\mathrm{H}^o_\rho[0,1]$ *for any* $\rho \in \mathcal{R}$*.* W *is a Brownian motion started at* 0*. Removing the term* $X_1\Lambda_1$ *in (14) gives a Brownian bridge* B*. The* ρ *sequential norms of* W *and* B *may be written as*

$$\|B\|_\rho^{\mathrm{seq}} = \sup_{j \geq 1} \frac{1}{\sqrt{2}\theta(2^j)} \max_{r \in D_j} |X_r|, \quad \|W\|_\rho^{\mathrm{seq}} = \max\Big\{\frac{|X_1|}{\rho(1)}, \|B\|_\rho^{\mathrm{seq}}\Big\}, \tag{15}$$

where

$$\theta(t) := t^{1/2}\rho(1/t), \quad t \geq 1.$$

We note that Condition (13) is optimal in view of the classical Lévy's result about the modulus of uniform continuity of the Brownian motion.

Proof. According to Prop. 3 c) in [14], the series (14) converges almost surely in the space $\mathrm{H}_\rho^o[0,1]$ if

$$\lim_{j\to\infty}\frac{1}{\rho(2^{-j})}\max_{r\in \mathrm{D}_j}|2^{-j/2}X_r| = 0, \text{ almost surely.} \tag{16}$$

A sufficient condition for the convergence (16) is that for every positive ε,

$$\sum_{j\geq 1}\mathbf{P}\Big\{\max_{r\in\mathrm{D}_j}|X_r|\geq \varepsilon\theta(2^j)\Big\} < \infty. \tag{17}$$

By identical distribution of the X_r's and the classical estimate $\mathbf{P}(|X_1|\geq t)\leq \exp(-t^2/2)$, (17) will follow in turn from

$$\sum_{j\geq 1}2^j\exp\left(\frac{-\varepsilon^2\theta^2(2^j)}{2}\right) < \infty,$$

which is easily deduced from Condition (13). Therefore (16) is satisfied and the series (14) converges almost surely in the space $\mathrm{H}_\rho^o[0,1]$, defining a mean zero Gaussian random element $W = X_1\Lambda_1 + B$ in $\mathrm{H}_\rho^o[0,1]$.

This convergence together with the obvious continuity of the λ_r's considered as linear functionals on $\mathrm{H}_\rho^o[0,1]$ legitimates the equality

$$\lambda_r(B) = \sum_{j=1}^\infty\sum_{r'\in\mathrm{D}_j}2^{-(j+1)/2}\lambda_r\big(X_{r'}\Lambda_{r'}\big), \quad \text{a.s.}$$

By Lemma 1 in [14] we get

$$\lambda_r\big(X_{r'}\Lambda_{r'}\big) = X_{r'}\lambda_r\big(\Lambda_{r'}\big) = X_{r'}\mathbf{1}_{\{r=r'\}}, \quad r, r'\in\mathrm{D}^*.$$

Hence for $r\in\mathrm{D}_j$ $(j\geq 1)$, $\lambda_r(B) = 2^{-(j+1)/2}X_r$ and (15) follows.

To complete the proof, it remains to check that W is a standard Brownian motion. It is convenient here to recast (2) as $2^{-(j+1)/2}\Lambda_r(t) = \langle H_r, \mathbf{1}_{[0,t]}\rangle$, where $\langle\ ,\ \rangle$ denotes the scalar product in the space $L^2[0,1]$. Now (14) implies clearly for each $t\in[0,1]$ that

$$W(t) = \sum_{r\in\mathrm{D}^*}X_r\langle H_r, \mathbf{1}_{[0,t]}\rangle,$$

where the series converges almost surely for each $t\in[0,1]$. So for any $0\leq s < t\leq 1$,

$$W(t) - W(s) = \sum_{r\in\mathrm{D}^*}X_r\langle H_r, \mathbf{1}_{(s,t]}\rangle$$

and this a.s. convergent series of independent mean zero Gaussian random variables converges also in quadratic mean, which legitimates the following covariance computation. For $0\leq s<t\leq 1$, $0\leq s'<t'\leq 1$, put

$$K(s,t,s',t') := \mathbf{E}\Big[\big(W(t)-W(s)\big)\big(W(t')-W(s')\big)\Big].$$

Noting that $\mathbf{E}\,(X_r X_{r'}) = \mathbf{1}_{\{r=r'\}}$ and using Parseval's identity for the Haar basis of $L^2[0,1]$ we obtain

$$K(s,t,s',t') = \sum_{r\in \mathrm{D}^*} \mathbf{E}\, X_r^2 \langle H_r, \mathbf{1}_{(s,t]}\rangle \langle H_r, \mathbf{1}_{(s',t']}\rangle = \langle \mathbf{1}_{(s,t]}, \mathbf{1}_{(s',t']}\rangle. \tag{18}$$

Whenever $(s,t]\cap(s',t'] = \emptyset$, (18) gives the independence of the Gaussian random variables $W(t)-W(s)$ and $W(t')-W(s')$, whence follows the independence of increments for the process W. Moreover (18) gives

$$K(s,t,s,t) = |t-s|.$$

As $W(0) = 0$, this achieves the identification of W as a version of the standard Brownian motion in the space $\mathrm{H}^o_\rho[0,1]$. □

4 Distributions of sequential norms

The distribution function of $\|B\|_\rho^{\mathrm{seq}}$ may be conveniently expressed in terms of the *error function*:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2)\,\mathrm{d}s = \mathbf{P}\big(|X_1| \le x\sqrt{2}\big), \quad x \ge 0.$$

The following asymptotic expansion will be useful

$$\operatorname{erf} x = 1 - \frac{1}{x\sqrt{\pi}} \exp(-x^2)\big(1 + O(x^{-2})\big), \quad x \to \infty. \tag{19}$$

Theorem 5. *Let* $c = \limsup_{j\to\infty} j^{1/2}/\theta(2^j)$.

i) If $c = \infty$ *then* $\|B\|_\rho^{\mathrm{seq}} = \|W\|_\rho^{\mathrm{seq}} = \infty$ *almost surely.*

ii) If $0 \le c < \infty$, *then* $\|B\|_\rho^{\mathrm{seq}}$ *and* $\|W\|_\rho^{\mathrm{seq}}$ *are almost surely finite and their distribution functions are given by*

$$\mathbf{P}\big(\|B\|_\rho^{\mathrm{seq}} \le x\big) = \prod_{j=1}^\infty \big\{\operatorname{erf}\big(\theta(2^j)x\big)\big\}^{2^{j-1}}, \quad x > 0$$

and

$$\mathbf{P}\big(\|W\|_\rho^{\mathrm{seq}} \le x\big) = \operatorname{erf}\big(2^{-1/2}\theta(1)x\big) \prod_{j=1}^\infty \big\{\operatorname{erf}\big(\theta(2^j)x\big)\big\}^{2^{j-1}}, \quad x > 0.$$

The support of these distributions is $[c\sqrt{\ln 2}, \infty)$.

Proof. Recalling (15), let us consider the non decreasing sequence of random variables $(M_J)_{J\geq 1}$ where

$$M_J := \max_{1\leq j\leq J} \frac{1}{\sqrt{2}\theta(2^j)} \max_{r\in D_j} |X_r|.$$

Considering $\|B\|_\rho^{\mathrm{seq}}$ as a random element in $[0,\infty]$, we have for each $x\in\mathbb{R}$,

$$\mathbf{P}(M_J\leq x)\downarrow \mathbf{P}(\|B\|_\rho^{\mathrm{seq}}\leq x) =: F_{B,\rho}(x), \quad (J\to\infty).$$

We have to deal with the two questions

a) Does there exists $x_0<\infty$ such that $F_{B,\rho}(x_0)>0$? In this case we will have $F_{B,\rho}>0$ on $[x_0,\infty)$.

b) $\lim_{x\to\infty} F_{B,\rho}(x)=1$?

Now by independence and equal distribution of the X_r's,

$$\mathbf{P}(M_J\leq x) = \prod_{j=1}^{J} \operatorname{erf}\big(x\theta(2^j)\big)^{2^{j-1}}. \tag{20}$$

Let us get rid first of the rough subcase of i) where $\theta(2^j)$ does not go to ∞ with j. Then there is a constant A and an infinite subset $\mathbb{J}_1$ of $\mathbb{N}$ such that $\theta(2^{j_k})\leq A$ for each $j\in\mathbb{J}_1$. Then noting that

$$0\leq M_J\leq \min_{j\in\mathbb{J}_1,\, j\leq J} \operatorname{erf}\big(Ax\big)^{2^{j-1}}$$

we directly see that $\lim_{J\to\infty} M_J=0$ for every $0\leq x<\infty$. So $F_{B,\rho}(x)=0$ for every $x\in\mathbb{R}$, which means exactly that $\|B\|_\rho^{\mathrm{seq}}=\infty$ with probability one. The same holds for W since $\|W\|_\rho^{\mathrm{seq}}\geq\|B\|_\rho^{\mathrm{seq}}$.

From now on we assume that $\lim_{j\to\infty}\theta(2^j)=\infty$. Taking logarithms in (20) and using (19), it is easily seen that a positive answer to question a) is equivalent to the convergence of the series $\sum_{j=1}^\infty u_j(x_0)$ where

$$u_j(x) := \frac{2^j}{x\theta(2^j)}\exp\big(-x^2\theta(2^j)^2\big) = \frac{1}{x}\exp\big(-x^2\theta(2^j)^2 + j\ln 2 - \ln\theta(2^j)\big).$$

If $c=\infty$, there is an infinite subset $\mathbb{J}_2$ of $\mathbb{N}$ such that $\lim_{\mathbb{J}_2\ni j\to\infty} j\theta(2^j)^{-2}=\infty$, so that $\lim_{\mathbb{J}_2\ni j\to\infty} u_j(x)=\infty$ and the series $\sum_{j=1}^\infty u_j(x)$ diverges for every $x\geq 0$. It follows that $F_{B,\rho}(x)=0$ for every $x\in\mathbb{R}$.

If $0\leq c<\infty$, we have for any positive ε some $J_0=J_0(\varepsilon)$ such that

$$\theta(2^j)\geq 1 \quad\text{and}\quad \frac{j}{\theta(2^j)^2}\leq (c+\varepsilon)^2, \quad j\geq J_0.$$

It follows that

$$u_j(x)\leq \frac{1}{x}\exp\Big\{-j\Big(\frac{x^2}{(c+\varepsilon)^2}-\ln 2\Big)\Big\}, \quad j\geq J_0,$$

so the series $\sum_{j=1}^{\infty} u_j(x)$ converges at geometric rate for $x > (c+\varepsilon)\sqrt{\ln 2}$. Now the answer to question b) is also positive noting that for $x > c\sqrt{\ln 2}$

$$F_{B,\rho}(x) = \prod_{j=1}^{\infty} \operatorname{erf}\bigl(\theta(2^j)x\bigr)^{2^{j-1}}$$

and applying the bounded convergence theorem (with $x \to \infty$) to the series

$$\ln F_{B,\rho}(x) = \sum_{j=1}^{\infty} 2^{j-1} \ln \operatorname{erf}\bigl(\theta(2^j)x\bigr).$$

To complete the proof, it remains to show that the support of the distribution of $\|B\|_\rho^{\mathrm{seq}}$ is $[c\sqrt{\ln 2}, \infty)$, for which it is enough to check that if $x < c\sqrt{\ln 2}$, $F_{B,\rho}(x) = 0$. The case $c = 0$ being obvious, let us assume $0 < c < \infty$. Having fixed $0 < x < c\sqrt{\ln 2}$, let us choose $0 < p < q < 1$, such that $x \le pc\sqrt{\ln 2}$. From the definition of c, $\theta(2^j)j^{-1/2}$ converges to $1/c$ along some subsequence. Hence we can find an *infinite* subset $\mathbb{J} \subset \mathbb{N}$ (depending on θ, x and q) such that

$$\theta(2^j) \le \frac{j^{1/2}}{qc}, \quad j \in \mathbb{J}.$$

Puting $r = p^2 q^{-2}$, we obtain the lower bound

$$u_j(x) = \frac{2^j}{x\theta(2^j)} \exp\bigl(-x^2\theta(2^j)^2\bigr) \ge \frac{1}{c\sqrt{r \ln 2}} j^{-1/2} 2^j 2^{-rj}, \quad j \in \mathbb{J}.$$

As $\mathbb{J}$ is infinite and $r < 1$, this is enough to entail $\sum_{j=1}^{\infty} u_j(x) = \infty$ and hence $F_{B,\rho}(x) = 0$. □

5 Practical computations

For the numerical computation of $F_{B,\rho}(x)$, we need to estimate the convergence rate of the corresponding infinite product. First we observe that for every positive x

$$\prod_{j=1}^{\infty} \bigl\{\operatorname{erf}\bigl(\theta(2^j)x\bigr)\bigr\}^{2^{j-1}} < \prod_{j=1}^{J} \bigl\{\operatorname{erf}\bigl(\theta(2^j)x\bigr)\bigr\}^{2^{j-1}} \quad J \ge 1.$$

So to control the relative error committed when approximating the infinite product by its J-th partial product, we simply need a practically computable lower bound $m(J)$ for

$$R_J := \prod_{j=J+1}^{\infty} \bigl\{\operatorname{erf}\bigl(\theta(2^j)x\bigr)\bigr\}^{2^{j-1}}.$$

Then we will have

$$m(J)\prod_{j=1}^{J}\big\{\operatorname{erf}\big(\theta(2^j)x\big)\big\}^{2^{j-1}} < F_{B,\rho}(x) < \prod_{j=1}^{J}\big\{\operatorname{erf}\big(\theta(2^j)x\big)\big\}^{2^{j-1}}. \qquad (21)$$

First we note the elementary estimate

$$\operatorname{erf} x \geq 1-\exp(-x^2), \quad x\geq 0, \qquad (22)$$

which is easily checked by the change of variable $s = x+u$ in the integral $1-\operatorname{erf} x = 2\pi^{-1/2}\int_x^\infty \exp(-s^2)\,\mathrm{d}s$ and the majorization of $\exp(-2xu)$ by 1 inside the new integral. (22) is better than (19) for small values of x. Combining (22) with the inequality

$$1-y \geq \exp(-2y), \quad 0\leq y\leq 0.7968,$$

we obtain provided that $\exp(-\theta(2^{J+1})^2x^2)\leq 0.7968$,

$$\big\{\operatorname{erf}\big(\theta(2^j)x\big)\big\}^{2^{j-1}} \geq \big\{1-\exp\big(-\theta(2^j)^2x^2\big)\big\}^{2^{j-1}} \geq \exp\Big(-2^j\exp\big(-\theta(2^j)^2x^2\big)\Big).$$

Hence we get for $x\theta(2^{J+1})\geq 0.4767$,

$$R_J \geq \exp\Big(-\sum_{j=J+1}^{\infty} 2^j\exp\big(-\theta(2^j)^2x^2\big)\Big).$$

Assuming that θ is non decreasing and comparing series and integral leads now to

$$R_J \geq \exp\Big(-2\int_{2^J}^{\infty}\exp\big(-x^2\theta(t)^2\big)\,\mathrm{d}t\Big), \quad \text{if} \quad x\theta(2^{J+1})\geq 0.4767. \qquad (23)$$

We are mainly interested in the two cases $\rho(h) = h^\alpha$ $(0<\alpha<1/2)$ and $\rho(h) = h^{1/2}\ln^\beta(\mathrm{e}^{2\beta}/h)$ $(\beta > 1/2)$. The constant $b = \mathrm{e}^{2\beta}$ is the smallest for which $\rho(h) = h^{1/2}\ln^\beta(b/h)$ fulfils Conditions (8) to (12).

Case $\rho(h) = h^\alpha$, *(*$0<\alpha<1/2$*).* We can express the integral

$$I(J) := \int_{2^J}^{\infty}\exp\big(-x^2\theta(t)^2\big)\,\mathrm{d}t$$

in terms of the Gamma distribution. Recall that the Gamma distribution with scale parameter λ and shape parameter q has density

$$g_{\lambda,q}(y) = \frac{\lambda^q}{\Gamma(q)}\mathrm{e}^{-\lambda y}y^{q-1}, \quad y\geq 0.$$

We write $G_{\lambda,q}$ the corresponding distribution function:

$$G_{\lambda,q}(x) = \int_0^x g_{\lambda,q}(y)\,\mathrm{d}y, \quad x\geq 0.$$

$G_{\lambda,q}(x)$ is computable by a software routine. In the current case,

$$\theta(t) = t^{1/p} \quad \text{where} \quad \frac{1}{p} = \frac{1}{2} - \alpha.$$

The change of variable $u = t^{2/p}$ gives now

$$I(J) = \int_{2^{2J/p}}^{\infty} \exp\bigl(-x^2 u\bigr) \frac{p}{2} u^{p/2-1}\,\mathrm{d}u = \frac{p\Gamma(p/2)}{2x^p} \int_{2^{2J/p}}^{\infty} g_{x^2,p/2}(u)\,\mathrm{d}u.$$

Case $\rho(h) = h^{1/2}\ln^{\beta}(\mathrm{e}^{2\beta}/h)$ *(*$\beta \geq 1/2$*).* Then

$$\theta(t)^2 = (2\beta + \ln t)^{2\beta}$$

and since $2\beta \geq 1$, we get for $t \geq 2^J$:

$$\theta(t)^2 \geq (2\beta)^{2\beta} + (\ln t)^{2\beta} \geq (2\beta)^{2\beta} + (J\ln 2)^{2\beta-1}\ln t.$$

Pluging this in $I(J)$ gives

$$I(J) \leq \exp\bigl(-x^2(2\beta)^{2\beta}\bigr) \int_{2^J}^{\infty} t^{-x^2(J\ln 2)^{2\beta-1}}\,\mathrm{d}t.$$

When $\beta > 1/2$, the support of the distribution of $\|B\|_{\rho}^{\mathrm{seq}}$ is $[0,\infty)$ and for any $x > 0$, we have some $J_0(x)$ such that $x^2(J\ln 2)^{2\beta-1} > 1$ so that the above integral converges. Clearly then

$$I(J) \leq \exp\bigl(-x^2(2\beta)^{2\beta}\bigr) \frac{2^{J(1-x^2(J\ln 2)^{2\beta-1})}}{x^2(J\ln 2)^{2\beta-1} - 1} \quad \text{if} \quad J > x^{-2/(2\beta-1)}(\ln 2)^{-1}.$$

In the critical case $\beta = 1/2$, we have $\limsup_{j\to\infty} j^{1/2}/\theta(2^j) = (\ln 2)^{-1/2}$, so the support of the distribution is $[1,\infty)$. Then for $x > 1$, we can use the same estimate as for the case $\beta > 1/2$. Here it takes the simpler form

$$I(J) \leq \frac{\exp(-x^2)}{x^2 - 1} 2^{J(1-x^2)}, \quad x > 1,\ J \geq 1.$$

Let us observe moreover that when $x > 1$, the preliminary condition $x\theta(2^{J+1}) \geq 0.4767$ writes here $x\bigl(1 + (J+1)\ln 2\bigr)^{1/2} \geq 0.4767$ and is automatically satisfied for $J \geq 1$.

Let us summarize now this discussion on the relative error.

Proposition 6. *The distribution function* $F_{B,\rho}(x)$ *of* $\|B\|_{\rho}^{\mathrm{seq}}$ *is always overestimated by the finite products* $\prod_{j=1}^{J}\bigl\{\mathrm{erf}\bigl(\theta(2^j)x\bigr)\bigr\}^{2^{j-1}}$. *Moreover the relative error of approximation is given when* θ *is non decreasing by*

$$m(J)\prod_{j=1}^{J}\bigl\{\mathrm{erf}\bigl(\theta(2^j)x\bigr)\bigr\}^{2^{j-1}} < F_{B,\rho}(x) < \prod_{j=1}^{J}\bigl\{\mathrm{erf}\bigl(\theta(2^j)x\bigr)\bigr\}^{2^{j-1}}$$

where

$$m(J) = \exp\Big(-2\int_{2^J}^{\infty} \exp\big(-x^2\theta(t)^2\big)\,\mathrm{d}t\Big), \quad \textit{for } x\theta(2^{J+1}) \geq 0.4767.$$

In the case $\rho(h) = h^{\alpha}$, *(*$0 < \alpha < 1/2$*), putting* $1/p = 1/2 - \alpha$,

$$m(J) = \exp\Big(-p\Gamma(p/2)\frac{1 - G_{x^2,\,p/2}\big(2^{2J/p}\big)}{x^p}\Big), \quad x2^{(J+1)/p} \geq 0.4767,$$

where $G_{\lambda,q}$ *is the Gamma d.f. with scale parameter* λ *and shape parameter* q.
In the case $\rho(h) = h^{1/2}\ln^{\beta}(\mathrm{e}^{2\beta}/h)$, $\beta > 1/2$,

$$m(J) \geq \exp\Big(-2\exp\big(-x^2(2\beta)^{2\beta}\big)\frac{2^{J(1-x^2(J\ln 2)^{2\beta-1})}}{x^2(J\ln 2)^{2\beta-1} - 1}\Big),$$

for

$$J > \max\Big(\frac{1}{x^{2/(2\beta-1)}\ln 2}; \frac{0.4767^{1/\beta}}{x^{1/\beta}\ln 2} - \frac{2\beta}{\ln 2} - 1\Big), \quad x > 0.$$

In the case $\rho(h) = h^{1/2}\ln^{1/2}(\mathrm{e}/h)$,

$$m(J) = \exp\Big(-\frac{2\exp(-x^2)}{x^2 - 1}2^{J(1-x^2)}\Big), \quad x > 1,\ J \geq 1.$$

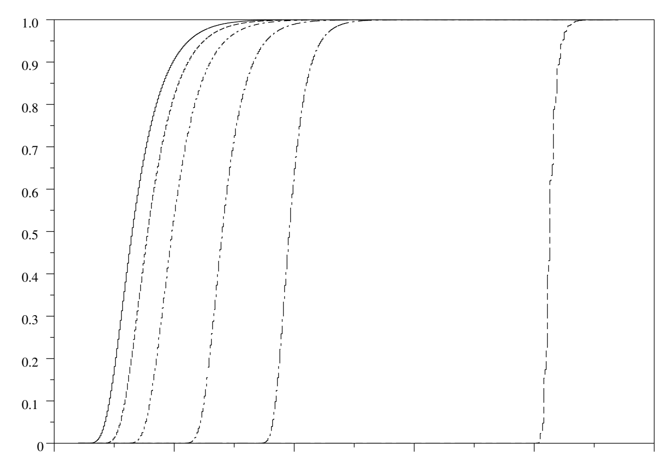

Figure 2: $F_{B,\rho}(x)$ for $\rho(h) = h^{\alpha}$, $\alpha = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.45,\ 0.49$

6 Tables

6.1 Tables of the values of $F_{B,\rho}(x)$ for $\rho(h) = h^{\alpha}$

	$\alpha = 0.05$									
x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.2	.00000	.00000	.00001	.00003	.00007	.00019	.00042	.00086	.00162	.00283
0.3	.00466	.00727	.01084	.01552	.02146	.02877	.03752	.04777	.05952	.07276
0.4	.08742	.10344	.12071	.13913	.15856	.17888	.19995	.22164	.24382	.26635
0.5	.28913	.31203	.33496	.35783	.38053	.40301	.42520	.44703	.46846	.48944
0.6	.50993	.52992	.54937	.56827	.58661	.60437	.62156	.63816	.65419	.66965
0.7	.68453	.69886	.71263	.72587	.73858	.75077	.76247	.77368	.78441	.79469
0.8	.80453	.81393	.82293	.83153	.83974	.84759	.85508	.86222	.86905	.87555
0.9	.88176	.88767	.89331	.89869	.90381	.90868	.91333	.91775	.92196	.92597
1.0	.92978	.93341	.93686	.94014	.94326	.94623	.94905	.95173	.95428	.95670
1.1	.95900	.96119	.96326	.96523	.96711	.96888	.97057	.97217	.97368	.97512
1.2	.97649	.97778	.97901	.98017	.98128	.98232	.98331	.98425	.98514	.98598
1.3	.98678	.98753	.98824	.98892	.98956	.99016	.99073	.99127	.99178	.99227
1.4	.99272	.99315	.99356	.99394	.99431	.99465	.99497	.99528	.99557	.99584
1.5	.99609	.99633	.99656	.99677	.99697	.99716	.99734	.99751	.99767	.99782
1.6	.99796	.99809	.99821	.99833	.99844	.99854	.99863	.99872	.99881	.99889
1.7	.99896	.99903	.99910	.99916	.99922	.99927	.99932	.99937	.99941	.99945
1.8	.99949	.99953	.99956	.99959	.99962	.99965	.99967	.99970	.99972	.99974
1.9	.99976	.99977	.99979	.99981	.99982	.99983	.99985	.99986	.99987	.99988
2.0	.99989	.99990	.99990	.99991	.99992	.99993	.99993	.99994	.99994	.99995
2.1	.99995	.99995	.99996	.99996	.99996	.99997	.99997	.99997	.99997	.99998

	$\alpha = 0.1$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.2	.00000	.00000	.00000	.00000	.00000	.00000	.00001	.00002	.00006	.00015
0.3	.00034	.00069	.00129	.00226	.00373	.00586	.00880	.01271	.01773	.02398
0.4	.03156	.04055	.05096	.06283	.07611	.09076	.10670	.12386	.14211	.16134
0.5	.18143	.20226	.22368	.24559	.26786	.29037	.31302	.33570	.35833	.38082
0.6	.40310	.42510	.44676	.46803	.48888	.50925	.52913	.54850	.56732	.58559
0.7	.60330	.62044	.63701	.65301	.66845	.68332	.69764	.71142	.72466	.73737
0.8	.74958	.76129	.77251	.78326	.79356	.80342	.81285	.82186	.83048	.83872
0.9	.84659	.85410	.86127	.86811	.87464	.88087	.88681	.89247	.89786	.90300
1.0	.90790	.91256	.91700	.92123	.92526	.92909	.93273	.93620	.93950	.94263
1.1	.94562	.94845	.95115	.95371	.95614	.95846	.96065	.96274	.96472	.96661
1.2	.96839	.97009	.97170	.97323	.97468	.97606	.97736	.97860	.97977	.98088
1.3	.98194	.98294	.98389	.98478	.98563	.98644	.98720	.98792	.98861	.98926
1.4	.98987	.99045	.99100	.99151	.99200	.99247	.99291	.99332	.99371	.99408
1.5	.99443	.99476	.99507	.99537	.99564	.99591	.99615	.99639	.99660	.99681
1.6	.99701	.99719	.99736	.99753	.99768	.99783	.99796	.99809	.99821	.99832
1.7	.99843	.99853	.99863	.99871	.99880	.99887	.99895	.99902	.99908	.99914
1.8	.99920	.99925	.99930	.99935	.99939	.99943	.99947	.99951	.99954	.99957
1.9	.99960	.99963	.99966	.99968	.99970	.99972	.99974	.99976	.99978	.99979
2.0	.99981	.99982	.99984	.99985	.99986	.99987	.99988	.99989	.99990	.99990
2.1	.99991	.99992	.99992	.99993	.99993	.99994	.99994	.99995	.99995	.99996
2.2	.99996	.99996	.99997	.99997	.99997	.99997	.99998	.99998	.99998	.99998
2.3	.99998	.99998	.99999	.99999	.99999	.99999	.99999	.99999	.99999	.99999

	$\alpha = 0.15$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.3	.00000	.00001	.00002	.00005	.00012	.00027	.00054	.00101	.00177	.00295
0.4	.00467	.00709	.01033	.01456	.01989	.02644	.03429	.04352	.05413	.06615
0.5	.07954	.09426	.11024	.12739	.14560	.16477	.18478	.20550	.22681	.24860
0.6	.27074	.29313	.31565	.33822	.36073	.38311	.40528	.42719	.44876	.46995
0.7	.49072	.51103	.53085	.55016	.56893	.58715	.60482	.62192	.63845	.65442
0.8	.66982	.68467	.69895	.71270	.72591	.73860	.75078	.76246	.77365	.78438
0.9	.79465	.80448	.81388	.82287	.83146	.83966	.84750	.85499	.86213	.86894
1.0	.87544	.88164	.88755	.89318	.89854	.90366	.90852	.91316	.91757	.92177
1.1	.92577	.92958	.93320	.93664	.93991	.94303	.94598	.94880	.95147	.95401
1.2	.95643	.95872	.96090	.96297	.96493	.96680	.96857	.97025	.97185	.97336
1.3	.97480	.97616	.97745	.97868	.97984	.98094	.98199	.98298	.98392	.98481
1.4	.98565	.98645	.98721	.98792	.98860	.98924	.98985	.99042	.99097	.99148
1.5	.99197	.99243	.99287	.99328	.99367	.99403	.99438	.99471	.99502	.99531
1.6	.99559	.99585	.99610	.99633	.99655	.99676	.99695	.99714	.99731	.99747
1.7	.99763	.99777	.99791	.99804	.99816	.99827	.99838	.99848	.99858	.99867
1.8	.99875	.99883	.99891	.99898	.99904	.99910	.99916	.99922	.99927	.99931
1.9	.99936	.99940	.99944	.99948	.99951	.99955	.99958	.99960	.99963	.99966
2.0	.99968	.99970	.99972	.99974	.99976	.99978	.99979	.99981	.99982	.99983
2.1	.99984	.99985	.99987	.99987	.99988	.99989	.99990	.99991	.99991	.99992
2.2	.99993	.99993	.99994	.99994	.99995	.99995	.99995	.99996	.99996	.99996
2.3	.99997	.99997	.99997	.99997	.99998	.99998	.99998	.99998	.99998	.99998

	$\alpha = 0.2$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.3	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00001	.00003
0.4	.00007	.00015	.00031	.00060	.00110	.00189	.00308	.00480	.00718	.01038
0.5	.01452	.01973	.02612	.03378	.04278	.05315	.06490	.07802	.09247	.10818
0.6	.12507	.14306	.16202	.18185	.20243	.22364	.24536	.26746	.28984	.31239
0.7	.33501	.35760	.38009	.40239	.42443	.44616	.46752	.48846	.50895	.52895
0.8	.54844	.56740	.58581	.60365	.62092	.63762	.65375	.66931	.68429	.69872
0.9	.71259	.72593	.73872	.75101	.76278	.77406	.78486	.79520	.80509	.81455
1.0	.82358	.83221	.84046	.84832	.85583	.86299	.86982	.87633	.88253	.88844
1.1	.89407	.89944	.90454	.90940	.91402	.91842	.92261	.92659	.93038	.93398
1.2	.93740	.94065	.94374	.94668	.94947	.95212	.95464	.95703	.95930	.96146
1.3	.96351	.96545	.96729	.96904	.97070	.97228	.97377	.97519	.97653	.97780
1.4	.97901	.98016	.98124	.98227	.98324	.98417	.98504	.98587	.98666	.98740
1.5	.98810	.98877	.98940	.98999	.99056	.99109	.99160	.99208	.99253	.99296
1.6	.99336	.99374	.99410	.99444	.99477	.99507	.99536	.99563	.99589	.99613
1.7	.99636	.99657	.99678	.99697	.99715	.99732	.99748	.99763	.99778	.99791
1.8	.99804	.99816	.99827	.99838	.99848	.99857	.99866	.99874	.99882	.99890
1.9	.99897	.99903	.99909	.99915	.99920	.99925	.99930	.99935	.99939	.99943
2.0	.99947	.99950	.99953	.99956	.99959	.99962	.99964	.99967	.99969	.99971
2.1	.99973	.99975	.99977	.99978	.99980	.99981	.99982	.99984	.99985	.99986
2.2	.99987	.99988	.99989	.99989	.99990	.99991	.99991	.99992	.99993	.99993
2.3	.99994	.99994	.99995	.99995	.99995	.99996	.99996	.99996	.99997	.99997
2.4	.99997	.99997	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99999

	$\alpha = 0.25$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.4	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00001	.00003	.00008
0.5	.00016	.00033	.00063	.00113	.00192	.00310	.00480	.00715	.01029	.01435
0.6	.01948	.02577	.03333	.04223	.05250	.06418	.07724	.09165	.10736	.12429
0.7	.14235	.16142	.18140	.20216	.22358	.24554	.26791	.29058	.31343	.33637
0.8	.35929	.38210	.40473	.42710	.44915	.47081	.49205	.51281	.53308	.55281
0.9	.57199	.59060	.60863	.62606	.64290	.65915	.67481	.68988	.70437	.71829
1.0	.73165	.74446	.75674	.76849	.77975	.79051	.80079	.81062	.82001	.82896
1.1	.83751	.84566	.85343	.86083	.86788	.87460	.88099	.88708	.89287	.89838
1.2	.90362	.90860	.91334	.91784	.92212	.92619	.93005	.93372	.93720	.94051
1.3	.94365	.94663	.94946	.95215	.95469	.95711	.95941	.96158	.96364	.96560
1.4	.96745	.96921	.97088	.97246	.97396	.97538	.97672	.97800	.97920	.98035
1.5	.98143	.98245	.98342	.98434	.98521	.98604	.98682	.98755	.98825	.98891
1.6	.98954	.99013	.99069	.99122	.99171	.99219	.99263	.99306	.99345	.99383
1.7	.99419	.99452	.99484	.99514	.99542	.99569	.99594	.99618	.99641	.99662
1.8	.99682	.99701	.99719	.99735	.99751	.99766	.99780	.99794	.99806	.99818
1.9	.99829	.99839	.99849	.99858	.99867	.99875	.99883	.99890	.99897	.99904
2.0	.99910	.99915	.99921	.99926	.99930	.99935	.99939	.99943	.99946	.99950
2.1	.99953	.99956	.99959	.99962	.99964	.99967	.99969	.99971	.99973	.99975
2.2	.99976	.99978	.99979	.99981	.99982	.99983	.99984	.99985	.99986	.99987
2.3	.99988	.99989	.99990	.99990	.99991	.99992	.99992	.99993	.99993	.99994
2.4	.99994	.99995	.99995	.99995	.99996	.99996	.99996	.99997	.99997	.99997
2.5	.99997	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99999	.99999

	$\alpha = 0.3$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.5	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00001
0.6	.00003	.00007	.00015	.00030	.00057	.00102	.00175	.00284	.00444	.00666
0.7	.00967	.01359	.01858	.02476	.03224	.04109	.05138	.06313	.07635	.09099
0.8	.10701	.12433	.14284	.16245	.18302	.20444	.22655	.24924	.27236	.29579
0.9	.31941	.34311	.36677	.39030	.41362	.43664	.45930	.48153	.50329	.52453
1.0	.54521	.56532	.58482	.60370	.62195	.63956	.65653	.67287	.68857	.70365
1.1	.71811	.73197	.74524	.75793	.77007	.78166	.79272	.80328	.81334	.82293
1.2	.83206	.84076	.84903	.85690	.86439	.87151	.87827	.88469	.89079	.89659
1.3	.90208	.90730	.91226	.91695	.92141	.92563	.92964	.93344	.93704	.94045
1.4	.94368	.94674	.94964	.95239	.95499	.95746	.95979	.96200	.96409	.96607
1.5	.96794	.96972	.97139	.97298	.97448	.97591	.97725	.97852	.97972	.98086
1.6	.98194	.98295	.98392	.98482	.98568	.98649	.98726	.98799	.98867	.98932
1.7	.98993	.99051	.99105	.99157	.99205	.99251	.99295	.99336	.99374	.99411
1.8	.99445	.99478	.99508	.99537	.99565	.99590	.99614	.99637	.99659	.99679
1.9	.99698	.99716	.99733	.99749	.99765	.99779	.99792	.99805	.99817	.99828
2.0	.99838	.99848	.99858	.99866	.99875	.99882	.99890	.99896	.99903	.99909
2.1	.99915	.99920	.99925	.99930	.99934	.99938	.99942	.99946	.99949	.99953
2.2	.99956	.99959	.99961	.99964	.99966	.99968	.99970	.99972	.99974	.99976
2.3	.99977	.99979	.99980	.99982	.99983	.99984	.99985	.99986	.99987	.99988
2.4	.99989	.99989	.99990	.99991	.99991	.99992	.99993	.99993	.99994	.99994
2.5	.99994	.99995	.99995	.99996	.99996	.99996	.99996	.99997	.99997	.99997
2.6	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99999	.99999

	$\alpha = 0.35$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.7	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00001	.00003	.00008
0.8	.00017	.00035	.00066	.00119	.00203	.00330	.00516	.00775	.01125	.01582
0.9	.02162	.02879	.03745	.04767	.05952	.07300	.08810	.10476	.12290	.14242
1.0	.16319	.18506	.20788	.23149	.25574	.28046	.30551	.33073	.35600	.38118
1.1	.40617	.43086	.45516	.47901	.50232	.52505	.54715	.56859	.58933	.60936
1.2	.62867	.64725	.66509	.68221	.69860	.71429	.72927	.74357	.75721	.77021
1.3	.78257	.79434	.80552	.81614	.82623	.83579	.84487	.85347	.86161	.86933
1.4	.87664	.88355	.89009	.89628	.90213	.90767	.91290	.91784	.92251	.92692
1.5	.93108	.93502	.93873	.94224	.94555	.94867	.95162	.95441	.95703	.95951
1.6	.96185	.96405	.96613	.96809	.96994	.97169	.97333	.97488	.97635	.97773
1.7	.97903	.98025	.98141	.98250	.98353	.98449	.98540	.98626	.98707	.98784
1.8	.98856	.98923	.98987	.99047	.99104	.99157	.99208	.99255	.99299	.99341
1.9	.99381	.99418	.99453	.99486	.99517	.99546	.99574	.99600	.99624	.99647
2.0	.99668	.99689	.99708	.99725	.99742	.99758	.99773	.99787	.99800	.99813
2.1	.99824	.99835	.99845	.99855	.99864	.99873	.99881	.99888	.99895	.99902
2.2	.99908	.99914	.99919	.99925	.99929	.99934	.99938	.99942	.99946	.99949
2.3	.99953	.99956	.99959	.99961	.99964	.99966	.99968	.99970	.99972	.99974
2.4	.99976	.99977	.99979	.99980	.99982	.99983	.99984	.99985	.99986	.99987
2.5	.99988	.99989	.99990	.99990	.99991	.99992	.99992	.99993	.99993	.99994
2.6	.99994	.99994	.99995	.99995	.99996	.99996	.99996	.99996	.99997	.99997
2.7	.99997	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99999

	$\alpha = 0.4$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.0	.00000	.00000	.00000	.00000	.00000	.00000	.00001	.00002	.00006	.00014
1.1	.00030	.00061	.00117	.00209	.00355	.00573	.00886	.01318	.01890	.02624
1.2	.03539	.04648	.05960	.07480	.09204	.11126	.13234	.15512	.17941	.20500
1.3	.23166	.25917	.28729	.31581	.34451	.37321	.40172	.42989	.45759	.48469
1.4	.51110	.53673	.56153	.58545	.60844	.63049	.65159	.67173	.69092	.70918
1.5	.72651	.74294	.75851	.77322	.78713	.80025	.81262	.82428	.83525	.84558
1.6	.85529	.86441	.87298	.88102	.88857	.89566	.90230	.90853	.91436	.91984
1.7	.92496	.92976	.93426	.93847	.94242	.94611	.94957	.95280	.95583	.95867
1.8	.96132	.96381	.96613	.96831	.97034	.97225	.97403	.97570	.97726	.97873
1.9	.98009	.98137	.98257	.98369	.98474	.98573	.98664	.98750	.98831	.98906
2.0	.98977	.99043	.99105	.99162	.99216	.99267	.99314	.99359	.99400	.99439
2.1	.99475	.99509	.99541	.99571	.99599	.99625	.99649	.99672	.99694	.99714
2.2	.99732	.99750	.99766	.99781	.99796	.99809	.99822	.99833	.99844	.99855
2.3	.99864	.99873	.99882	.99889	.99897	.99904	.99910	.99916	.99922	.99927
2.4	.99932	.99936	.99941	.99945	.99948	.99952	.99955	.99958	.99961	.99964
2.5	.99966	.99968	.99970	.99973	.99974	.99976	.99978	.99979	.99981	.99982
2.6	.99983	.99984	.99986	.99987	.99987	.99988	.99989	.99990	.99991	.99991
2.7	.99992	.99992	.99993	.99993	.99994	.99994	.99995	.99995	.99995	.99996
2.8	.99996	.99996	.99997	.99997	.99997	.99997	.99998	.99998	.99998	.99998
2.9	.99998	.99998	.99998	.99999	.99999	.99999	.99999	.99999	.99999	.99999

	$\alpha = 0.45$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.7	.00001	.00004	.00013	.00034	.00082	.00177	.00350	.00642	.01101	.01778
1.8	.02722	.03976	.05572	.07526	.09839	.12496	.15468	.18715	.22189	.25838
1.9	.29608	.33447	.37307	.41142	.44916	.48594	.52152	.55569	.58830	.61925
2.0	.64847	.67595	.70168	.72570	.74805	.76880	.78801	.80575	.82212	.83719
2.1	.85105	.86378	.87545	.88616	.89596	.90493	.91314	.92064	.92750	.93376
2.2	.93948	.94471	.94948	.95383	.95781	.96144	.96475	.96777	.97053	.97305
2.3	.97535	.97745	.97937	.98112	.98272	.98419	.98552	.98674	.98786	.98888
2.4	.98981	.99066	.99144	.99216	.99281	.99341	.99395	.99445	.99491	.99533
2.5	.99572	.99607	.99639	.99669	.99696	.99721	.99744	.99765	.99784	.99801
2.6	.99817	.99832	.99846	.99858	.99870	.99880	.99890	.99899	.99907	.99915
2.7	.99921	.99928	.99934	.99939	.99944	.99948	.99952	.99956	.99960	.99963
2.8	.99966	.99969	.99971	.99973	.99976	.99978	.99979	.99981	.99983	.99984
2.9	.99985	.99986	.99987	.99988	.99989	.99990	.99991	.99992	.99992	.99993
3.0	.99994	.99994	.99995	.99995	.99995	.99996	.99996	.99996	.99997	.99997
3.1	.99997	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99999	.99999

	$\alpha = 0.46$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.9	.00000	.00000	.00000	.00000	.00000	.00002	.00007	.00021	.00057	.00139
2.0	.00302	.00596	.01084	.01834	.02913	.04377	.06266	.08597	.11362	.14532
2.1	.18058	.21880	.25926	.30124	.34404	.38698	.42948	.47105	.51127	.54984
2.2	.58651	.62114	.65363	.68395	.71211	.73815	.76215	.78419	.80437	.82282
2.3	.83963	.85494	.86885	.88147	.89291	.90327	.91264	.92111	.92877	.93568
2.4	.94192	.94756	.95264	.95723	.96137	.96510	.96846	.97150	.97424	.97671
2.5	.97894	.98095	.98277	.98441	.98589	.98722	.98843	.98952	.99051	.99140
2.6	.99220	.99293	.99359	.99419	.99473	.99521	.99566	.99606	.99642	.99675
2.7	.99705	.99731	.99756	.99778	.99798	.99817	.99833	.99848	.99862	.99874
2.8	.99886	.99896	.99905	.99914	.99921	.99928	.99935	.99940	.99946	.99951
2.9	.99955	.99959	.99963	.99966	.99969	.99972	.99974	.99976	.99978	.99980
3.0	.99982	.99984	.99985	.99986	.99988	.99989	.99990	.99990	.99991	.99992
3.1	.99993	.99993	.99994	.99994	.99995	.99995	.99996	.99996	.99996	.99997
3.2	.99997	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99999	.99999

	$\alpha = 0.47$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
2.3	.00001	.00003	.00013	.00044	.00123	.00299	.00643	.01249	.02218	.03647
2.4	.05606	.08139	.11247	.14895	.19008	.23486	.28227	.33126	.38080	.42991
2.5	.47778	.52384	.56762	.60887	.64727	.68280	.71543	.74523	.77238	.79690
2.6	.81903	.83892	.85676	.87271	.88698	.89969	.91099	.92104	.92997	.93791
2.7	.94494	.95118	.95671	.96160	.96594	.96979	.97320	.97621	.97888	.98125
2.8	.98335	.98521	.98686	.98832	.98961	.99076	.99178	.99268	.99349	.99420
2.9	.99483	.99539	.99589	.99634	.99673	.99709	.99740	.99768	.99793	.99815
3.0	.99834	.99852	.99868	.99882	.99894	.99905	.99915	.99924	.99932	.99939
3.1	.99945	.99951	.99956	.99961	.99965	.99968	.99971	.99974	.99977	.99979
3.2	.99981	.99983	.99985	.99987	.99988	.99989	.99990	.99991	.99992	.99993
3.3	.99994	.99994	.99995	.99995	.99996	.99996	.99997	.99997	.99997	.99997
3.4	.99998	.99998	.99998	.99998	.99998	.99999	.99999	.99999	.99999	.99999

	$\alpha = 0.48$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
2.8	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00002
2.9	.00009	.00037	.00129	.00390	.00923	.01857	.03458	.06011	.09193	.13152
3.0	.17834	.23708	.29432	.35099	.41206	.47581	.53169	.58213	.62860	.67975
3.1	.71823	.75392	.78482	.81683	.84014	.86158	.87972	.89646	.91211	.92413
3.2	.93423	.94303	.95206	.95845	.96433	.96908	.97334	.97746	.98055	.98315
3.3	.98540	.98771	.98932	.99078	.99201	.99312	.99415	.99495	.99561	.99618
3.4	.99672	.99720	.99757	.99789	.99816	.99845	.99864	.99882	.99897	.99910
3.5	.99924	.99934	.99942	.99950	.99956	.99963	.99967	.99972	.99975	.99978
3.6	.99981	.99984	.99986	.99988	.99989	.99991	.99992	.99993	.99994	.99995
3.7	.99995	.99996	.99996	.99997	.99997	.99998	.99998	.99998	.99998	.99999

	$\alpha = 0.49$									
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
4.0	.00000	.00000	.00050	.00078	.00221	.02333	.02894	.04752	.15377	.17390
4.1	.22609	.39724	.43118	.61994	.63204	.65775	.78808	.79899	.82429	.88802
4.2	.89402	.90844	.94243	.94563	.95319	.97084	.97245	.97632	.98541	.98614
4.3	.98812	.99268	.99305	.99405	.99633	.99652	.99702	.99816	.99826	.99850
4.4	.99908	.99913	.99925	.99954	.99956	.99962	.99977	.99978	.99981	.99988
4.5	.99989	.99990	.99994	.99994	.99995	.99996	.99997	.99997	.99998	.99999

6.2 Tables of the values of $F_{B,\rho}(x)$ for $\rho(h) = h^{1/2} \ln^{\beta}(\mathrm{e}^{2\beta}/h)$

	$\alpha = 0.5,\ \beta = 3$									
x	.0000	.0001	.0002	.0003	.0004	.0005	.0006	.0007	.0008	.0009
.001	.00000	.00000	.00000	.00001	.00015	.00098	.00399	.01149	.02584	.04866
.002	.08034	.12012	.16646	.21746	.27121	.32598	.38035	.43322	.48380	.53157
.003	.57622	.61760	.65570	.69058	.72237	.75125	.77739	.80099	.82227	.84141
.004	.85860	.87402	.88785	.90022	.91129	.92119	.93003	.93792	.94496	.95123
.005	.95682	.96180	.96623	.97017	.97367	.97678	.97954	.98198	.98415	.98608
.006	.98778	.98928	.99061	.99178	.99282	.99373	.99453	.99523	.99585	.99640
.007	.99687	.99729	.99765	.99797	.99825	.99849	.99870	.99888	.99904	.99918
.008	.99930	.99940	.99949	.99956	.99963	.99968	.99973	.99977	.99981	.99984
.009	.99986	.99989	.99990	.99992	.99993	.99994	.99995	.99996	.99997	.99997
.010	.99998	.99998	.99998	.99999	.99999	.99999	.99999	.99999	1.0000	1.0000

	$\alpha = 0.5,\ \beta = 2.5$									
x	.0000	.0001	.0002	.0003	.0004	.0005	.0006	.0007	.0008	.0009
.005	.00000	.00000	.00000	.00001	.00001	.00003	.00006	.00011	.00020	.00034
.006	.00055	.00087	.00133	.00197	.00282	.00394	.00537	.00716	.00936	.01201
.007	.01515	.01882	.02306	.02789	.03332	.03938	.04607	.05338	.06132	.06986
.008	.07901	.08873	.09899	.10978	.12106	.13279	.14495	.15749	.17038	.18359
.009	.19707	.21079	.22472	.23881	.25305	.26739	.28180	.29626	.31074	.32521
.010	.33965	.35404	.36835	.38257	.39668	.41066	.42449	.43818	.45169	.46503
.011	.47818	.49113	.50389	.51643	.52876	.54087	.55276	.56442	.57586	.58707
.012	.59805	.60880	.61932	.62961	.63967	.64950	.65912	.66850	.67767	.68662
.013	.69535	.70387	.71218	.72028	.72818	.73588	.74339	.75070	.75782	.76476
.014	.77151	.77809	.78449	.79072	.79679	.80269	.80843	.81401	.81945	.82473
.015	.82987	.83487	.83973	.84445	.84904	.85351	.85784	.86206	.86616	.87014
.016	.87401	.87777	.88142	.88497	.88842	.89177	.89502	.89818	.90125	.90423
.017	.90713	.90994	.91267	.91532	.91789	.92039	.92282	.92517	.92746	.92968
.018	.93183	.93392	.93595	.93792	.93983	.94169	.94349	.94524	.94693	.94858
.019	.95018	.95173	.95323	.95469	.95611	.95748	.95881	.96010	.96136	.96257
.020	.96375	.96490	.96601	.96708	.96813	.96914	.97012	.97107	.97200	.97289
.021	.97376	.97460	.97542	.97621	.97697	.97772	.97844	.97914	.97981	.98047
.022	.98110	.98172	.98232	.98290	.98346	.98400	.98453	.98504	.98553	.98601
.023	.98647	.98692	.98736	.98778	.98819	.98858	.98897	.98934	.98970	.99004
.024	.99038	.99071	.99102	.99133	.99162	.99191	.99218	.99245	.99271	.99296
.025	.99320	.99344	.99367	.99389	.99410	.99430	.99450	.99470	.99488	.99506
.026	.99524	.99540	.99557	.99572	.99588	.99602	.99616	.99630	.99643	.99656
.027	.99669	.99681	.99692	.99703	.99714	.99724	.99735	.99744	.99754	.99763
.028	.99771	.99780	.99788	.99796	.99803	.99811	.99818	.99825	.99831	.99838
.029	.99844	.99850	.99855	.99861	.99866	.99871	.99876	.99881	.99885	.99890
.030	.99894	.99898	.99902	.99906	.99910	.99913	.99917	.99920	.99923	.99926
.031	.99929	.99932	.99935	.99937	.99940	.99942	.99944	.99947	.99949	.99951
.032	.99953	.99955	.99957	.99958	.99960	.99962	.99963	.99965	.99966	.99968
.033	.99969	.99970	.99972	.99973	.99974	.99975	.99976	.99977	.99978	.99979
.034	.99980	.99981	.99982	.99982	.99983	.99984	.99984	.99985	.99986	.99986
.035	.99987	.99988	.99988	.99989	.99989	.99990	.99990	.99991	.99991	.99991
.036	.99992	.99992	.99992	.99993	.99993	.99993	.99994	.99994	.99994	.99995
.037	.99995	.99995	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99997

	$\alpha = 0.5,\ \beta = 2$									
x	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.02	.00000	.00002	.00017	.00078	.00255	.00655	.01402	.02605	.04335	.06617
.03	.09426	.12701	.16359	.20306	.24446	.28692	.32966	.37204	.41354	.45374
.04	.49237	.52920	.56412	.59706	.62800	.65695	.68396	.70909	.73243	.75407
.05	.77409	.79259	.80967	.82542	.83994	.85330	.86560	.87690	.88730	.89684
.06	.90561	.91366	.92105	.92783	.93405	.93975	.94498	.94977	.95415	.95817
.07	.96185	.96522	.96831	.97113	.97371	.97607	.97823	.98020	.98200	.98365
.08	.98515	.98652	.98777	.98891	.98995	.99089	.99175	.99254	.99325	.99390
.09	.99449	.99503	.99551	.99596	.99636	.99672	.99705	.99735	.99761	.99786
.10	.99808	.99828	.99846	.99862	.99876	.99889	.99901	.99912	.99921	.99930
.11	.99938	.99944	.99951	.99956	.99961	.99965	.99969	.99973	.99976	.99979
.12	.99981	.99983	.99985	.99987	.99989	.99990	.99991	.99992	.99993	.99994
.13	.99995	.99995	.99996	.99997	.99997	.99997	.99998	.99998	.99998	.99999

	$\alpha = 0.5,\ \beta = 1.5$									
x	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.07	.00000	.00000	.00000	.00001	.00002	.00003	.00007	.00013	.00024	.00041
.08	.00068	.00108	.00164	.00242	.00346	.00481	.00654	.00869	.01132	.01446
.09	.01817	.02248	.02741	.03300	.03924	.04616	.05375	.06199	.07088	.08040
.10	.09052	.10121	.11243	.12416	.13635	.14896	.16196	.17531	.18895	.20286
.11	.21699	.23130	.24576	.26034	.27500	.28970	.30442	.31913	.33381	.34842
.12	.36296	.37739	.39170	.40588	.41990	.43375	.44743	.46091	.47420	.48728
.13	.50015	.51279	.52521	.53741	.54937	.56109	.57258	.58384	.59485	.60563
.14	.61617	.62648	.63655	.64639	.65600	.66538	.67453	.68347	.69218	.70068
.15	.70897	.71705	.72492	.73259	.74007	.74735	.75444	.76134	.76807	.77461
.16	.78098	.78718	.79322	.79909	.80480	.81036	.81576	.82102	.82614	.83111
.17	.83595	.84065	.84523	.84968	.85400	.85820	.86229	.86626	.87012	.87388
.18	.87752	.88107	.88452	.88787	.89112	.89428	.89736	.90034	.90325	.90607
.19	.90881	.91147	.91406	.91657	.91901	.92138	.92369	.92593	.92810	.93022
.20	.93227	.93426	.93620	.93808	.93991	.94169	.94341	.94509	.94671	.94829
.21	.94983	.95132	.95277	.95417	.95554	.95687	.95816	.95941	.96062	.96180
.22	.96295	.96406	.96514	.96619	.96721	.96820	.96916	.97009	.97099	.97187
.23	.97273	.97355	.97436	.97514	.97590	.97663	.97735	.97804	.97871	.97937
.24	.98000	.98061	.98121	.98179	.98235	.98290	.98343	.98394	.98444	.98493
.25	.98539	.98585	.98629	.98672	.98714	.98754	.98793	.98831	.98868	.98904
.26	.98939	.98972	.99005	.99036	.99067	.99097	.99126	.99154	.99181	.99207
.27	.99232	.99257	.99281	.99304	.99327	.99349	.99370	.99390	.99410	.99429
.28	.99448	.99466	.99484	.99501	.99517	.99533	.99548	.99563	.99578	.99592
.29	.99606	.99619	.99631	.99644	.99656	.99667	.99679	.99689	.99700	.99710
.30	.99720	.99729	.99739	.99748	.99756	.99765	.99773	.99780	.99788	.99795
.31	.99802	.99809	.99816	.99822	.99828	.99834	.99840	.99846	.99851	.99857
.32	.99862	.99867	.99871	.99876	.99880	.99885	.99889	.99893	.99896	.99900
.33	.99904	.99907	.99911	.99914	.99917	.99920	.99923	.99926	.99928	.99931
.34	.99934	.99936	.99938	.99941	.99943	.99945	.99947	.99949	.99951	.99953
.35	.99955	.99956	.99958	.99960	.99961	.99963	.99964	.99965	.99967	.99968
.36	.99969	.99970	.99972	.99973	.99974	.99975	.99976	.99977	.99978	.99978
.37	.99979	.99980	.99981	.99982	.99982	.99983	.99984	.99984	.99985	.99986
.38	.99986	.99987	.99987	.99988	.99988	.99989	.99989	.99990	.99990	.99990
.39	.99991	.99991	.99992	.99992	.99992	.99993	.99993	.99993	.99993	.99994
.40	.99994	.99994	.99995	.99995	.99995	.99995	.99995	.99996	.99996	.99996
.41	.99996	.99996	.99996	.99997	.99997	.99997	.99997	.99997	.99997	.99997
.42	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998
.43	.99998	.99998	.99999	.99999	.99999	.99999	.99999	.99999	.99999	.99999

	$\alpha = 0.5,\ \beta = 1$									
x	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.24	.00000	.00000	.00000	.00000	.00000	.00000	.00001	.00001	.00002	.00002
.25	.00003	.00005	.00007	.00009	.00012	.00016	.00021	.00028	.00036	.00046
.26	.00058	.00072	.00089	.00110	.00134	.00162	.00195	.00233	.00276	.00326
.27	.00382	.00445	.00515	.00594	.00682	.00778	.00884	.01001	.01128	.01266
.28	.01415	.01577	.01750	.01936	.02135	.02347	.02572	.02811	.03063	.03329
.29	.03609	.03903	.04210	.04532	.04867	.05217	.05579	.05956	.06345	.06748
.30	.07164	.07592	.08033	.08486	.08951	.09428	.09916	.10414	.10923	.11443
.31	.11972	.12511	.13058	.13615	.14180	.14752	.15332	.15920	.16514	.17114
.32	.17720	.18332	.18949	.19571	.20197	.20828	.21462	.22100	.22740	.23383
.33	.24029	.24677	.25326	.25977	.26629	.27282	.27936	.28589	.29243	.29897
.34	.30550	.31202	.31854	.32504	.33153	.33801	.34446	.35090	.35732	.36371
.35	.37008	.37642	.38274	.38903	.39528	.40151	.40770	.41386	.41998	.42606
.36	.43211	.43813	.44410	.45003	.45592	.46177	.46758	.47335	.47907	.48475
.37	.49039	.49598	.50153	.50703	.51249	.51790	.52327	.52859	.53386	.53908
.38	.54426	.54940	.55448	.55952	.56451	.56946	.57436	.57921	.58401	.58877
.39	.59348	.59815	.60276	.60734	.61186	.61634	.62078	.62516	.62951	.63381
.40	.63806	.64227	.64643	.65055	.65463	.65866	.66265	.66660	.67050	.67436
.41	.67818	.68196	.68569	.68939	.69304	.69666	.70023	.70376	.70726	.71071
.42	.71413	.71750	.72084	.72414	.72741	.73063	.73382	.73698	.74010	.74318
.43	.74622	.74923	.75221	.75515	.75806	.76094	.76378	.76659	.76936	.77210
.44	.77482	.77750	.78014	.78276	.78535	.78790	.79043	.79293	.79540	.79783
.45	.80024	.80263	.80498	.80730	.80960	.81187	.81412	.81634	.81853	.82069
.46	.82283	.82495	.82704	.82910	.83114	.83316	.83515	.83712	.83906	.84098
.47	.84288	.84476	.84661	.84844	.85025	.85204	.85381	.85556	.85728	.85899
.48	.86067	.86234	.86398	.86561	.86721	.86880	.87036	.87191	.87344	.87496
.49	.87645	.87793	.87938	.88083	.88225	.88366	.88505	.88642	.88778	.88912
.50	.89044	.89175	.89305	.89433	.89559	.89684	.89807	.89929	.90049	.90168
.51	.90286	.90402	.90516	.90630	.90742	.90852	.90962	.91070	.91177	.91282
.52	.91386	.91489	.91591	.91692	.91791	.91889	.91986	.92082	.92177	.92270
.53	.92363	.92454	.92545	.92634	.92722	.92809	.92895	.92980	.93064	.93147
.54	.93229	.93310	.93390	.93470	.93548	.93625	.93701	.93777	.93851	.93925
.55	.93998	.94070	.94141	.94211	.94281	.94349	.94417	.94484	.94550	.94616
.56	.94680	.94744	.94807	.94869	.94931	.94992	.95052	.95111	.95170	.95228
.57	.95286	.95342	.95398	.95454	.95508	.95562	.95616	.95668	.95721	.95772
.58	.95823	.95873	.95923	.95972	.96021	.96069	.96116	.96163	.96209	.96255
.59	.96300	.96345	.96389	.96432	.96476	.96518	.96560	.96602	.96643	.96683
.60	.96724	.96763	.96802	.96841	.96879	.96917	.96954	.96991	.97028	.97064
.61	.97099	.97135	.97169	.97204	.97238	.97271	.97304	.97337	.97370	.97401
.62	.97433	.97464	.97495	.97526	.97556	.97586	.97615	.97644	.97673	.97701
.63	.97729	.97757	.97784	.97811	.97838	.97865	.97891	.97917	.97942	.97967
.64	.97992	.98017	.98041	.98065	.98089	.98112	.98135	.98158	.98181	.98203
.65	.98225	.98247	.98269	.98290	.98311	.98332	.98352	.98373	.98393	.98413
.66	.98432	.98452	.98471	.98490	.98508	.98527	.98545	.98563	.98581	.98598
.67	.98616	.98633	.98650	.98667	.98683	.98699	.98716	.98732	.98747	.98763
.68	.98778	.98794	.98809	.98823	.98838	.98853	.98867	.98881	.98895	.98909
.69	.98923	.98936	.98949	.98963	.98976	.98988	.99001	.99014	.99026	.99038
.70	.99050	.99062	.99074	.99086	.99097	.99109	.99120	.99131	.99142	.99153
.71	.99163	.99174	.99184	.99195	.99205	.99215	.99225	.99235	.99245	.99254
.72	.99264	.99273	.99282	.99291	.99300	.99309	.99318	.99327	.99335	.99344
.73	.99352	.99361	.99369	.99377	.99385	.99393	.99400	.99408	.99416	.99423
.74	.99431	.99438	.99445	.99452	.99459	.99466	.99473	.99480	.99487	.99493
.75	.99500	.99506	.99513	.99519	.99525	.99531	.99538	.99544	.99549	.99555
.76	.99561	.99567	.99572	.99578	.99584	.99589	.99594	.99600	.99605	.99610
.77	.99615	.99620	.99625	.99630	.99635	.99640	.99644	.99649	.99654	.99658
.78	.99663	.99667	.99672	.99676	.99680	.99684	.99689	.99693	.99697	.99701
.79	.99705	.99709	.99713	.99716	.99720	.99724	.99728	.99731	.99735	.99738

	$\alpha = 0.5,\ \beta = 1$									
x	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.80	.99742	.99745	.99749	.99752	.99755	.99759	.99762	.99765	.99768	.99771
.81	.99774	.99777	.99780	.99783	.99786	.99789	.99792	.99795	.99798	.99800
.82	.99803	.99806	.99808	.99811	.99813	.99816	.99818	.99821	.99823	.99826
.83	.99828	.99830	.99833	.99835	.99837	.99840	.99842	.99844	.99846	.99848
.84	.99850	.99852	.99854	.99856	.99858	.99860	.99862	.99864	.99866	.99868
.85	.99870	.99871	.99873	.99875	.99877	.99878	.99880	.99882	.99883	.99885
.86	.99887	.99888	.99890	.99891	.99893	.99894	.99896	.99897	.99899	.99900
.87	.99901	.99903	.99904	.99906	.99907	.99908	.99910	.99911	.99912	.99913
.88	.99915	.99916	.99917	.99918	.99919	.99920	.99922	.99923	.99924	.99925
.89	.99926	.99927	.99928	.99929	.99930	.99931	.99932	.99933	.99934	.99935
.90	.99936	.99937	.99938	.99939	.99939	.99940	.99941	.99942	.99943	.99944
.91	.99944	.99945	.99946	.99947	.99948	.99948	.99949	.99950	.99951	.99951
.92	.99952	.99953	.99953	.99954	.99955	.99955	.99956	.99957	.99957	.99958
.93	.99959	.99959	.99960	.99960	.99961	.99962	.99962	.99963	.99963	.99964
.94	.99964	.99965	.99965	.99966	.99966	.99967	.99967	.99968	.99968	.99969
.95	.99969	.99970	.99970	.99971	.99971	.99971	.99972	.99972	.99973	.99973
.96	.99974	.99974	.99974	.99975	.99975	.99975	.99976	.99976	.99977	.99977
.97	.99977	.99978	.99978	.99978	.99979	.99979	.99979	.99980	.99980	.99980
.98	.99980	.99981	.99981	.99981	.99982	.99982	.99982	.99982	.99983	.99983
.99	.99983	.99984	.99984	.99984	.99984	.99985	.99985	.99985	.99985	.99985
1.00	.99986	.99986	.99986	.99986	.99987	.99987	.99987	.99987	.99987	.99988
1.01	.99988	.99988	.99988	.99988	.99989	.99989	.99989	.99989	.99989	.99989
1.02	.99990	.99990	.99990	.99990	.99990	.99990	.99991	.99991	.99991	.99991
1.03	.99991	.99991	.99991	.99992	.99992	.99992	.99992	.99992	.99992	.99992
1.04	.99992	.99993	.99993	.99993	.99993	.99993	.99993	.99993	.99993	.99993
1.05	.99994	.99994	.99994	.99994	.99994	.99994	.99994	.99994	.99994	.99994
1.06	.99995	.99995	.99995	.99995	.99995	.99995	.99995	.99995	.99995	.99995
1.07	.99995	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99996
1.08	.99996	.99996	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997
1.09	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997
1.10	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99998	.99998
1.11	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998
1.12	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998
1.13	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99999	.99999	.99999

	$\alpha = 0.5,\ \beta = 0.9$									
x	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.31	.00000	.00000	.00000	.00001	.00001	.00001	.00002	.00003	.00004	.00005
.32	.00007	.00009	.00011	.00015	.00019	.00025	.00031	.00039	.00049	.00060
.33	.00074	.00090	.00109	.00131	.00156	.00185	.00218	.00256	.00298	.00346
.34	.00399	.00459	.00525	.00597	.00677	.00765	.00861	.00965	.01078	.01200
.35	.01331	.01472	.01623	.01784	.01956	.02138	.02331	.02535	.02751	.02977
.36	.03215	.03465	.03726	.03998	.04282	.04578	.04884	.05203	.05532	.05873
.37	.06224	.06587	.06960	.07343	.07737	.08141	.08555	.08979	.09412	.09854
.38	.10306	.10765	.11233	.11710	.12194	.12685	.13184	.13690	.14202	.14721
.39	.15246	.15776	.16312	.16853	.17399	.17950	.18505	.19063	.19626	.20192
.40	.20761	.21333	.21908	.22485	.23065	.23646	.24229	.24813	.25399	.25985
.41	.26573	.27161	.27749	.28337	.28925	.29513	.30101	.30688	.31275	.31860
.42	.32444	.33027	.33609	.34189	.34768	.35344	.35919	.36492	.37063	.37631
.43	.38197	.38760	.39321	.39880	.40435	.40988	.41538	.42085	.42629	.43169
.44	.43707	.44241	.44772	.45300	.45825	.46345	.46863	.47377	.47887	.48394
.45	.48897	.49397	.49893	.50385	.50873	.51358	.51839	.52316	.52790	.53260
.46	.53726	.54188	.54646	.55101	.55551	.55998	.56441	.56881	.57316	.57748
.47	.58176	.58600	.59020	.59437	.59850	.60259	.60664	.61066	.61464	.61858
.48	.62249	.62636	.63019	.63399	.63775	.64148	.64517	.64883	.65245	.65603
.49	.65958	.66310	.66658	.67003	.67344	.67682	.68017	.68348	.68677	.69001
.50	.69323	.69642	.69957	.70269	.70578	.70884	.71187	.71486	.71783	.72077
.51	.72367	.72655	.72940	.73222	.73501	.73777	.74050	.74321	.74589	.74854
.52	.75116	.75375	.75632	.75887	.76138	.76387	.76634	.76877	.77119	.77358
.53	.77594	.77828	.78059	.78288	.78515	.78739	.78961	.79181	.79398	.79613
.54	.79826	.80036	.80245	.80451	.80655	.80857	.81057	.81254	.81450	.81644
.55	.81835	.82025	.82212	.82398	.82581	.82763	.82943	.83120	.83296	.83471
.56	.83643	.83813	.83982	.84149	.84314	.84478	.84639	.84799	.84958	.85114
.57	.85269	.85423	.85575	.85725	.85873	.86020	.86166	.86310	.86452	.86593
.58	.86733	.86871	.87007	.87142	.87276	.87408	.87539	.87669	.87797	.87924
.59	.88049	.88173	.88296	.88418	.88538	.88657	.88775	.88892	.89007	.89121
.60	.89234	.89346	.89456	.89566	.89674	.89781	.89887	.89992	.90096	.90198
.61	.90300	.90401	.90500	.90599	.90696	.90793	.90888	.90983	.91076	.91169
.62	.91260	.91351	.91440	.91529	.91617	.91704	.91789	.91875	.91959	.92042
.63	.92124	.92206	.92287	.92367	.92446	.92524	.92601	.92678	.92754	.92829
.64	.92903	.92976	.93049	.93121	.93192	.93263	.93332	.93402	.93470	.93537
.65	.93604	.93671	.93736	.93801	.93865	.93929	.93991	.94054	.94115	.94176
.66	.94236	.94296	.94355	.94414	.94471	.94529	.94585	.94641	.94697	.94752
.67	.94806	.94860	.94913	.94966	.95018	.95070	.95121	.95171	.95221	.95271
.68	.95320	.95368	.95416	.95464	.95511	.95557	.95603	.95649	.95694	.95739
.69	.95783	.95827	.95870	.95913	.95955	.95997	.96039	.96080	.96120	.96161
.70	.96201	.96240	.96279	.96318	.96356	.96394	.96431	.96468	.96505	.96541
.71	.96577	.96613	.96648	.96683	.96718	.96752	.96785	.96819	.96852	.96885
.72	.96917	.96949	.96981	.97013	.97044	.97074	.97105	.97135	.97165	.97195
.73	.97224	.97253	.97281	.97310	.97338	.97366	.97393	.97420	.97447	.97474
.74	.97500	.97527	.97552	.97578	.97603	.97628	.97653	.97678	.97702	.97726
.75	.97750	.97774	.97797	.97820	.97843	.97865	.97888	.97910	.97932	.97954
.76	.97975	.97996	.98017	.98038	.98059	.98079	.98100	.98120	.98139	.98159
.77	.98178	.98197	.98216	.98235	.98254	.98272	.98290	.98309	.98326	.98344
.78	.98362	.98379	.98396	.98413	.98430	.98446	.98463	.98479	.98495	.98511
.79	.98527	.98542	.98558	.98573	.98588	.98603	.98618	.98633	.98647	.98662
.80	.98676	.98690	.98704	.98718	.98731	.98745	.98758	.98771	.98784	.98797
.81	.98810	.98823	.98835	.98848	.98860	.98872	.98884	.98896	.98908	.98920
.82	.98931	.98943	.98954	.98965	.98976	.98987	.98998	.99009	.99020	.99030
.83	.99041	.99051	.99061	.99071	.99081	.99091	.99101	.99111	.99120	.99130
.84	.99139	.99148	.99158	.99167	.99176	.99185	.99193	.99202	.99211	.99219
.85	.99228	.99236	.99244	.99253	.99261	.99269	.99277	.99285	.99292	.99300

	$\alpha = 0.5,\ \beta = 0.9$									
x	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.86	.99308	.99315	.99323	.99330	.99337	.99344	.99352	.99359	.99366	.99373
.87	.99379	.99386	.99393	.99400	.99406	.99413	.99419	.99426	.99432	.99438
.88	.99444	.99450	.99456	.99462	.99468	.99474	.99480	.99486	.99491	.99497
.89	.99502	.99508	.99513	.99519	.99524	.99529	.99535	.99540	.99545	.99550
.90	.99555	.99560	.99565	.99569	.99574	.99579	.99584	.99588	.99593	.99597
.91	.99602	.99606	.99611	.99615	.99619	.99624	.99628	.99632	.99636	.99640
.92	.99644	.99648	.99652	.99656	.99660	.99664	.99667	.99671	.99675	.99679
.93	.99682	.99686	.99689	.99693	.99696	.99700	.99703	.99706	.99710	.99713
.94	.99716	.99719	.99723	.99726	.99729	.99732	.99735	.99738	.99741	.99744
.95	.99747	.99750	.99753	.99755	.99758	.99761	.99764	.99766	.99769	.99772
.96	.99774	.99777	.99779	.99782	.99784	.99787	.99789	.99792	.99794	.99797
.97	.99799	.99801	.99803	.99806	.99808	.99810	.99812	.99815	.99817	.99819
.98	.99821	.99823	.99825	.99827	.99829	.99831	.99833	.99835	.99837	.99839
.99	.99841	.99842	.99844	.99846	.99848	.99850	.99851	.99853	.99855	.99857
1.00	.99858	.99860	.99862	.99863	.99865	.99866	.99868	.99869	.99871	.99873
1.01	.99874	.99876	.99877	.99878	.99880	.99881	.99883	.99884	.99885	.99887
1.02	.99888	.99889	.99891	.99892	.99893	.99895	.99896	.99897	.99898	.99900
1.03	.99901	.99902	.99903	.99904	.99905	.99907	.99908	.99909	.99910	.99911
1.04	.99912	.99913	.99914	.99915	.99916	.99917	.99918	.99919	.99920	.99921
1.05	.99922	.99923	.99924	.99925	.99926	.99927	.99927	.99928	.99929	.99930
1.06	.99931	.99932	.99933	.99933	.99934	.99935	.99936	.99937	.99937	.99938
1.07	.99939	.99940	.99940	.99941	.99942	.99943	.99943	.99944	.99945	.99945
1.08	.99946	.99947	.99947	.99948	.99949	.99949	.99950	.99950	.99951	.99952
1.09	.99952	.99953	.99953	.99954	.99955	.99955	.99956	.99956	.99957	.99957
1.10	.99958	.99958	.99959	.99959	.99960	.99960	.99961	.99961	.99962	.99962
1.11	.99963	.99963	.99964	.99964	.99965	.99965	.99966	.99966	.99966	.99967
1.12	.99967	.99968	.99968	.99968	.99969	.99969	.99970	.99970	.99970	.99971
1.13	.99971	.99972	.99972	.99972	.99973	.99973	.99973	.99974	.99974	.99974
1.14	.99975	.99975	.99975	.99976	.99976	.99976	.99977	.99977	.99977	.99977
1.15	.99978	.99978	.99978	.99979	.99979	.99979	.99979	.99980	.99980	.99980
1.16	.99980	.99981	.99981	.99981	.99981	.99982	.99982	.99982	.99982	.99983
1.17	.99983	.99983	.99983	.99983	.99984	.99984	.99984	.99984	.99985	.99985
1.18	.99985	.99985	.99985	.99986	.99986	.99986	.99986	.99986	.99986	.99987
1.19	.99987	.99987	.99987	.99987	.99987	.99988	.99988	.99988	.99988	.99988
1.20	.99988	.99989	.99989	.99989	.99989	.99989	.99989	.99989	.99990	.99990
1.21	.99990	.99990	.99990	.99990	.99990	.99991	.99991	.99991	.99991	.99991
1.22	.99991	.99991	.99991	.99992	.99992	.99992	.99992	.99992	.99992	.99992
1.23	.99992	.99992	.99992	.99993	.99993	.99993	.99993	.99993	.99993	.99993
1.24	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99994	.99994	.99994
1.25	.99994	.99994	.99994	.99994	.99994	.99995	.99995	.99995	.99995	.99995
1.26	.99995	.99995	.99995	.99995	.99995	.99995	.99995	.99995	.99995	.99995
1.27	.99996	.99996	.99996	.99996	.99996	.99996	.99996	.99996	.99996	.99996
1.28	.99996	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997	.99997
1.29	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997
1.30	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997	.99997
1.31	.99997	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998
1.32	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998
1.33	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998	.99998
1.34	.99998	.99998	.99998	.99998	.99998	.99998	.99999	.99999	.99999	.99999

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