PUB. IRMA, LILLE 2011
Vol. 71, N° VIII

Functional limit theorems for sums of nearly non stationary processes*

Jurgita Markevičiūtė [a, c]
Alfredas Račkauskas [a, b]
Charles Suquet [c]

Abstract

We study some Hölderian functional central limit theorems for the polygonal line partial sums process build on a first order autoregressive process $y_{n,k} = \phi_n y_{n,k-1} + \epsilon_k$ with ϕ_n converging to 1 and i.i.d. centered square integrable innovations. In the case where $\phi_n = \mathrm{e}^{\gamma/n}$ with is a negative constant γ, the limiting process is an integrated Ornstein-Uhlenbeck process. In the case where $\phi_n = 1 - \gamma_n/n$, where γ_n goes to infinity slower than n, we discuss the convergence in Hölder topology to Brownian motion in terms of the rate of γ_n and of the integrability of the ϵ_k's.

Résumé

Nous étudions certains théorèmes limite hölderiens pour le processus polygonal de sommes partielles bâti sur un processus autorégressif d'ordre un $y_{n,k} = \phi_n y_{n,k-1} + \epsilon_k$ avec ϕ_n convergeant vers 1 et des innovations ϵ_k i.i.d. centrées et de carré intégrable. Dans le cas où $\phi_n = \mathrm{e}^{\gamma/n}$ avec une constante négative γ, le processus limite est un Ornstein-Uhlenbeck intégré. Dans le cas où $\phi_n = 1 - \gamma_n/n$, où γ_n tend vers l'infini plus lentement que n, nous discutons la convergence hölderienne vers le mouvement brownien en fonction de la vitesse de γ_n et du degré d'intégrabilité des innovations.

MSC 2000 subject classifications. 60F17, 62M10.

Key words and phrases. Autoregressive process, Brownian motion, functional central limit theorem, Hölder space, integrated Ornstein-Uhlenbeck process, maximal inequality, partial sums, polygonal line.

---

*Preprint

[a]Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-2006 Vilnius, Lithuania.

[b]Institute of Mathematics and Informatics, Akademijos str. 4, LT-08663, Vilnius, Lithuania.

[c]Laboratoire P. Painlevé, UMR 8524 CNRS Université Lille I, Bât M2, Cité Scientifique, F-59655 Villeneuve d'Ascq Cedex, France.

1 Introduction

Suppose we have a first-order autoregressive process $\{y_{n,k} | 1 \le k \le n; n \ge 1\}$ given by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k \tag{1}$$

where $\phi_n \to 1$, as $n \to \infty$, $\{\varepsilon_k, k \ge 1\}$ is a sequence of independent, identically distributed random variables with $\mathbb{E}\varepsilon_k = 0$ and $\mathbb{E}\varepsilon_k^2 = 1$, n is a sample size and initialization $y_{n0} = 0$. Despite the fact that $y_{n,k}$ is a triangular array, for simplicity, we shall omit the index n and we shall write $y_k = \phi_n y_{k-1} + \varepsilon_k$. The process $\{y_k, k \ge 0\}$ when $\phi_n \to 1$, $n \to \infty$ is called *nearly nonstationary process.*

When $\phi_n = \phi$ does not depend on n, we have classical autoregressive model. If $|\phi| < 1$ then we have stationary autoregressive process, if $\phi = 1$ then we have a random walk and for $|\phi| > 1$ we have an explosive autoregressive process.

The coefficient ϕ_n is estimated with least squares estimator using observations $(y_1, y_2, \dots y_n)$:

$$\widehat{\phi}_n = \frac{\sum_{j=1}^n y_j y_{j-1}}{\sum_{j=1}^n y_{j-1}^2}.$$

The difference between real coefficient and its estimate is defined by:

$$\widehat{\phi}_n - \phi_n = \frac{\sum_{j=1}^n \varepsilon_j y_{j-1}}{\sum_{j=1}^n y_{j-1}^2}. \tag{2}$$

Our aim in this paper is to investigate convergence of polygonal line processes built on the y_k's:

$$S_n^{\mathrm{st}}(t) := \sum_{k=1}^{[nt]} y_{k-1}, \quad t \in [0,1], \tag{3}$$

$$S_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} y_{k-1} + (nt - [nt]) y_{[nt]}, \quad t \in [0,1]. \tag{4}$$

Remark 1.1. The definition of the partial sums process S_n^{st} and S_n^{pl} are quite unusual with a general term y_{k-1} were one would expect y_k. However, asymptotic results in theorem 3.2 remains true and with y_{k-1} replaced by y_k, see Remark 3.3.

We shall investigate asymptotic behaviour of S_n^{st} and S_n^{pl} in various function spaces considering the following two cases

- *Case 1*: $\phi_n = \mathrm{e}^{\gamma/n}$ (γ is a negative constant), where the limiting process is integrated Ornstein-Uhlenbeck process;
- *Case 2*: $\phi_n = 1 - \frac{\gamma_n}{n}$, $\gamma_n \to \infty$ slower than n, we discuss the convergence to Brownian motion.

For the case 1 simulation studies of Evans and Savin ([3], [4]) showed that the coefficient estimator and the t test in a nearly nonstationary $AR(1)$ process have statistical properties closer to asymptotic theory for random walk than to classical asymptotic theory that applies for stationary time series. This holds even in moderately large sample ($n = 50, 100$).

Phillips [8] and Cumberland and Sykes [2] found that normalized processes $n^{-1/2} y_{[nt]}$ converges weakly to an Ornstein-Uhlenbeck process defined by Itô stochastic differential equation

$$\mathrm{d}U_\gamma(t) = -\gamma U_\gamma(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t). \tag{5}$$

The asymptotic behaviour of the nearly non stationary process $\{y_k\}$ when $\phi_n = \mathrm{e}^{\gamma/n}$ involves the process

$$U_\gamma(t) = \int_0^t \mathrm{e}^{(t-s)\gamma}\,\mathrm{d}W(s) \quad 0 \le t \le 1. \tag{6}$$

In particular U_γ is an Ornstein-Uhlenbeck process generated by the stochastic differential equation (5) with initialization $U_\gamma(0) = 0$. It is worth noticing here that ([8]) :

$$U_\gamma(t) = W(t) + \gamma \int_0^t \mathrm{e}^{(t-s)\gamma} W(s)\,\mathrm{d}s \tag{7}$$

and

$$U_\gamma^2(1) = 1 + 2\gamma \int_0^1 U_\gamma^2(t)\,\mathrm{d}t + 2\int_0^1 U_\gamma(t)\,\mathrm{d}W(t). \tag{8}$$

Phillips [8] have found the asymptotic results for moments of nearly non stationary processes (see 4.1).

The case 2 was investigated by Giraitis and Phillips [5]. They also found the asymptotic results for moments of such process (see 4.1).

The step process of i.i.d. random variables is

$$W_n^{\mathrm{st}}(t) = \sum_{j=1}^{[nt]} \varepsilon_j. \tag{9}$$

The random elements W_n^{st} and S_n^{st} lie in Skorohod space $\mathrm{D}[0,1]$ endowed with the uniform norm

$$\|f\|_\infty = \sup_{0\le t\le 1} |f(t)|\,,$$

for any $f \in \mathrm{D}[0,1]$. Also, the polygonal line processes W_n^{pl}

$$W_n^{\mathrm{pl}}(t) = \sum_{j=1}^{[nt]} \varepsilon_j + (nr - [nt])\varepsilon_{[nt]+1} \tag{10}$$

and S_n^{pl} can be viewed as a random element either in $\mathrm{C}[0,1]$ or in $\mathrm{H}_\alpha^o[0,1]$. Continuous function space $\mathrm{C}[0,1]$ is endowed with the uniform norm

$$\|f\|_\infty = \sup_{0\le t\le 1} |f(t)| \quad f \in \mathrm{C}[0,1].$$

Separable Hölder space $\mathrm{H}_\alpha^o[0,1]$

$$\mathrm{H}_\alpha^o[0,1] := \left\{ f \in C[0,1] : \lim_{\delta\to 0} \omega_\alpha(f,\delta) = 0 \right\}, \quad \alpha \in (0,1)$$

is endowed with the norm

$$\|f\|_\alpha := |f(0)| + \omega(f,1),$$

where

$$\omega_\alpha(f,\delta) := \sup_{\substack{s,t\in[0,1]\\ 0<t-s<\delta}} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

From Donsker's theorem [1] we have

$$n^{-1/2}\sigma^{-1}W_n^{\mathrm{st}} \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} W, \tag{11}$$

where $\{W(r), 0 < r < 1\}$ is a standard Wiener process. The classical Donsker-Prohorov invariance principle says that

$$n^{-1/2}\sigma^{-1}W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W. \tag{12}$$

Here "$\xrightarrow[n\to\infty]{\mathrm{E}}$" signifies convergence in distribution in the metric space E. Accordingly, the classical convergence in distribution of a sequence of random variables is denoted by "$\xrightarrow[n\to\infty]{\mathbb{R}}$" and convergence in probability is denoted by $\xrightarrow[n\to\infty]{\mathrm{P}}$.

The weak convergence of a sequence of stochastic processes in some functions space F provides results about the asymptotic distribution of functionals of the paths which are continuous with respect to the topology of F. Since the Hölder spaces are topologically embedded in $\mathrm{C}[0,1]$ and in $\mathrm{D}[0,1]$, they support more continuous functionals. From this point of view, the alternative framework of Hölder spaces gives functional limit theorems of a wider scope (see more in [7]). From Račkauskas and Suquet [9] we have that for $0 < \alpha < 1/2$

$$n^{-1/2}\sigma^{-1}W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^0[0,1]} W \tag{13}$$

if and only if

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \ge t) = 0. \tag{14}$$

The paper is organized as follows: section 2 is devoted to case 1; section 3 is devoted to case 2; section 4 contains the more technical parts of some proofs.

2 Functional limit theorems in Case 1

We study the case where $\phi_n = \mathrm{e}^{\gamma/n}$. Note that instead of putting any direct assumption on the ε_j's, we assume rather some functional weak convergence of W_n^{pl} to the Brownian motion W. This extends the scope of the result far beyond the case where the ε_j's are i.i.d. (for some Hölderian invariance principles, in the case of weakly dependent random variables, see Hamadouche [6]). In what follows, we assume for notational simplicity that $\sigma = 1$.

Theorem 2.1. *Suppose that y_k is generated by* (1)*, $\phi_n = \mathrm{e}^{\gamma/n}$ and that the polygonal line $n^{-1/2}W_n^{\mathrm{pl}}$ converges weakly to the standard Brownian motion W in* $\mathrm{C}[0,1]$ *or in* $\mathrm{H}_\alpha^o[0,1]$ *for some $0 < \alpha < 1/2$. Then $n^{-3/2}S_n^{\mathrm{pl}}$ converges weakly in the same space to the integrated Ornstein-Uhlenbeck process J defined by:*

$$J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s, \quad 0 \le t \le 1, \tag{15}$$

where $U_\gamma(s) = \int_0^s \mathrm{e}^{\gamma(s-r)}\,\mathrm{d}W(r)$.

Proof. We will give the proof for the space $\mathrm{H}_\alpha^o[0,1]$. The adaptation to the simpler case of $\mathrm{C}[0,1]$ follows essentially by taking $\alpha = 0$.

The tightness of $n^{-1/2}W_n^{\mathrm{pl}}$ in $\mathrm{H}_\alpha^o[0,1]$ implies that

$$\left\| n^{-1/2} W_n^{\mathrm{pl}} \right\|_\infty \quad \text{is stochastically bounded} \tag{16}$$

and

$$\omega_\alpha\Big(n^{-1/2}W_n^{\mathrm{pl}}, \frac{1}{n}\Big) \xrightarrow[n\to\infty]{\mathrm{P}} 0, \tag{17}$$

see Theorem 13 in [10]. It is useful here to represent the polygonal line π_n with vertices $(l/n, V_l)$, $0 \le l \le 1$, $V_0 = 0$, under the form:

$$\pi_n(t) = (1 - \{nt\})V_{[nt]} + \{nt\}\,V_{[nt]+1}, \quad 0 \le t \le 1, \tag{18}$$

where $\{nt\} = nt - [nt]$ is the fractional part of nt. The Hölder norm of such a line satisfies

$$\|\pi_n\|_\alpha \le 2n^\alpha \max_{1 \le l \le n} |V_l|\,. \tag{19}$$

Indeed it is well known that this norm is reached at two vertices, so

$$\|\pi_n\|_\alpha = \max_{0 \le j < k \le n} \frac{|V_k - V_j|}{(\frac{k}{n} - \frac{j}{n})^\alpha},$$

whence (19) follows. As a consequence of (19), if we approximate each V_l by some $\widetilde{V_l}$ in such a way that $|V_l - \widetilde{V_l}| = o_P(n^{-\alpha})$, *uniformly* in $1 \le l \le n$, then the corresponding polygonal line $\widetilde{\pi_n}$ satisfies $\|\pi_n - \widetilde{\pi_n}\|_\alpha = o_P(1)$. Applying repeatedly this simple argument to the polygonal line $n^{-3/2}S_n^{\mathrm{pl}}$ with vertices

$(l/n, Y_l)$ where $Y_l = n^{-3/2}\sum_{k=1}^{l} y_{k-1}$ and with the help of (16) and (17), one reduce the problem to the convergence of the polygonal line π_n with vertices $(l/n, V_l)$, where:

$$V_l = \int_0^{l/n} n^{-1/2} W_n^{\mathrm{pl}}(s)\,\mathrm{d}s + \gamma \int_0^{l/n}\int_0^s \mathrm{e}^{\gamma(s-r)} n^{-1/2} W_n^{\mathrm{pl}}(r)\,\mathrm{d}r, \tag{20}$$

see Section 4 for the details.

Next we note that π_n is exactly the linear interpolation at the points $t_{n,l} = l/n$ of the random function:

$$J_n(t) := \int_0^t n^{-1/2} W_n^{\mathrm{pl}}(s)\,\mathrm{d}s + \gamma \int_0^t \int_0^s \mathrm{e}^{\gamma(s-r)} n^{-1/2} W_n^{\mathrm{pl}}(r)\,\mathrm{d}r.$$

By an elementary chaining argument, the interpolation error is controlled by

$$\|J_n - \pi_n\|_\alpha \le 4\omega_\alpha\Big(J_n, \frac{1}{n}\Big),$$

which converges in probability to zero, provided that J_n converges weakly in $\mathrm{H}_\alpha^o[0,1]$.

Finally, the problem is reduced in checking that J_n converges weakly to J in $\mathrm{H}_\alpha^o[0,1]$. As the functional

$$x \longmapsto \int_0^\bullet x(s)\,\mathrm{d}s + \gamma \int_0^\bullet \int_0^s \mathrm{e}^{\gamma(s-r)} x(r)\,\mathrm{d}r$$

is continuous on $\mathrm{H}_\alpha^o[0,1]$, this last convergence follows from the convergence of $n^{-1/2}W_n^{\mathrm{pl}}$ to W. □

Corollary 2.2. *When the ε_k's are i.i.d. and centered, the weak convergence of $n^{-3/2}S_n^{\mathrm{pl}}$ to J holds*

- *in* $\mathrm{C}[0,1]$ *provided that* $\mathbb{E}\varepsilon_1^2 < \infty$*;*
- *in* $\mathrm{H}_\alpha^o[0,1]$ *under condition* (14)*.*

3 Functional limit theorems in Case 2

In this section we will investigate the summation processes S_n^{st} and S_n^{pl} built on the y_k's, defined by (3) and (4), where $\phi_n = 1 - \gamma_n/n$ and $\gamma_n \to \infty$ slower than n.

A key point in all the following limit theorems is to have a good control on the asymptotic behaviour of $\max_{1\le k\le n}|y_k|$. This is provided by the following lemma which may be of independent interest.

Lemma 3.1. *Let $p \ge 2$. Assume that the innovations ε_k satisfy*

$$\begin{matrix} \lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0, & \text{if } p > 2 \\ \mathbb{E}\varepsilon_0^2 < \infty & \text{if } p = 2 \end{matrix} \tag{21}$$

For $p \geq 2$, put $\alpha = 1/2 - 1/p$. Then

$$n^{-1/2}\gamma_n^{\alpha} \max_{1\leq k\leq n} |y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{22}$$

Our main results in C[0,1] and D[0,1] cases.

Theorem 3.2. *Suppose y_k is generated by (1) and $\phi_n = 1 - \gamma_n/n$, where γ_n is non negative and goes to infinity slower than n. Assume also that the innovations $\{\varepsilon_k\}$ are i.i.d. with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = 1$. Denote by $W = \{W(t), 0 \leq t \leq 1\}$ a standard Wiener process. Then the following convergences hold.*

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{st}} \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} W, \tag{23}$$

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W. \tag{24}$$

Proof. For proving (23), it is enough to show that

$$\Delta_n = \left\| \frac{1-\phi_n}{n^{1/2}} S_n^{\mathrm{st}} - n^{-1/2} W_n^{\mathrm{st}} \right\|_\infty \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Note first that

$$\begin{aligned}
\Delta_n &= \sup_{0\leq t\leq 1} \left| \frac{1-\phi_n}{n^{1/2}} \sum_{j=1}^{[nt]} y_{j-1} - n^{-1/2} \sum_{j=1}^{[nt]} \varepsilon_j \right| \\
&= n^{-1/2} \max_{1\leq k\leq n} \left| (1-\phi_n) \sum_{j=1}^{k} y_{j-1} - \sum_{j=1}^{k} \varepsilon_j \right|.
\end{aligned}$$

For every $k \geq 1$, it follows from (1) that $\sum_{j=1}^k y_j = \phi_n \sum_{j=1}^k y_{j-1} + \sum_{j=1}^k \varepsilon_j$, whence

$$(1-\phi_n) \sum_{j=1}^{k} y_{j-1} = y_0 - y_k + \sum_{j=1}^{k} \varepsilon_j \tag{25}$$

and

$$\Delta_n = n^{-1/2} \max_{1\leq k\leq n} |y_0 - y_k| \leq n^{-1/2} |y_0| + n^{-1/2} \max_{1\leq k\leq n} |y_k|.$$

As y_0 does not depend on n, $n^{-1/2}y_0$ goes to 0 in probability. By the special case $p = 2$ in Lemma 3.1, the same convergence holds true for $n^{-1/2}\max_{1\leq k\leq n}|y_k|$ and hence for Δ_n.

To check the convergence (24), we use the same method, replacing Δ_n by $\Delta_n' = \|\xi_n\|_\infty$, where

$$\xi_n = \frac{1-\phi_n}{n^{1/2}} S_n^{\mathrm{pl}} - n^{-1/2} W_n^{\mathrm{pl}}$$

We observe then that ξ_n is a polygonal line with vertices at the points $t_{n,k} = k/n$, $0 \leq k \leq n$. The supremum norm of such a line is reached at one of its vertices. As S_n^{pl} and W_n^{pl} coincide with S_n^{st} and W_n^{st} respectively at the $t_{n,k}$'s, then clearly $\Delta_n' = \Delta_n$ and the proof is complete. □

Remark 3.3. If we modify the definition of S_n^{st} in

$$S_n^{\mathrm{st}}(t) = \sum_{k=1}^{[nt]} y_k$$

then Δ_n becomes

$$\Delta_n = \phi_n n^{-1/2} \max_{1\le k\le n} |y_0 - y_k| \le \phi_n n^{-1/2} |y_0| + \phi_n n^{-1/2} \max_{1\le k\le n} |y_k| .$$

As $0 < \phi_n \le 1$ for all n and $\phi_n \to 1$, $n \to \infty$ so $\{\phi_n\}$ is a bounded sequence and asymptotic result remains the same. For S_n^{pl} changed to

$$S_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} y_k + (nt - [nt])y_{[nt]+1}, \quad t \in [0,1],$$

we have the same limit process because of the considerations in the theorem 3.2. Further in this paper we shall use definitions 3 and 4.

Finally, we can find the limit proces of S_n^{pl} in the $\mathrm{H}_\beta^0[0,1]$.

Theorem 3.4. *Suppose y_k is generated by* (1) *and $\phi_n = 1 - \gamma_n/n$, where γ_n is non negative and goes to infinity slower than n. Assume also that the innovations $\{\varepsilon_k\}$ are i.i.d. and satisfy condition* (21) *for some $p > 2$. Put $\alpha = \frac{1}{2} - \frac{1}{p}$. Then for $0 < \beta \le \alpha$,*

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\beta^0[0,1]} W, \tag{26}$$

where W is a standard Wiener process if

$$\lim_{n\to\infty} \gamma_n n^{-\frac{\beta}{\alpha}} = \infty. \tag{27}$$

Proof. By [9], condition (21) gives the weak convergence of W_n^{pl}, defined by (10), to the standard Brownian motion in the space $\mathrm{H}_\alpha^o[0,1]$. By continuous embedding of Hölder spaces, the same convergence remains true in $H_\beta^0[0,1]$ for $0 < \beta \le \alpha$. Therefore it is enough to show that

$$D_{n,\beta} := \left\| n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}} \right\|_\beta \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Using the definition of the process and properties of Hölder norm we obtain

$$\begin{aligned}\left\| n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}} \right\|_\beta &\le \max_{1\le j<k\le n} \frac{\left|n^{-1/2}(y_k - y_j)\right|}{|k/n - j/n|^\beta} \\ &\le 2n^{\beta-\frac{1}{2}} \max_{1\le 2\le n} |y_k| .\end{aligned}$$

By lemma 3.1 $\max_{1\le k\le n} |y_k| = o_P(n^{1/2}\gamma_n^{-\alpha})$, so the above convergence is satisfied provided that

$$\lim_{n\to\infty} \frac{n^\beta}{\gamma_n^\alpha} = 0$$

which is equivalent to our assumption (27). □

4 Annex

4.1 Auxiliary results

Lemma 4.1. (Phillips [8]). *If* $\{y_k, k \geq 0\}$ *is a nearly non stationary process generated by* (1) *and* $\phi_n = \mathrm{e}^{\gamma/n}$, $\{\varepsilon_k, k \geq 0\}$ *are i.i.d. random variables with* $\mathbb{E}\varepsilon_0 = 0$ *and* $\mathbb{E}\varepsilon_0^2 = \sigma^2$, *then :*

$$n^{-1/2} y_{[nt]} \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} \sigma U_\gamma(t), \tag{28}$$

$$n^{-3/2} \sum_{j=1}^{n} y_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma \int_0^1 U_\gamma(r)\,\mathrm{d}r, \tag{29}$$

$$n^{-2} \sum_{j=1}^{n} y_j^2 \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 U_\gamma^2(r)\,\mathrm{d}r, \tag{30}$$

$$n^{-1} \sum_{j=1}^{n} y_{j-1}\varepsilon_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 U_\gamma(r)\,\mathrm{d}W(r). \tag{31}$$

Lemma 4.2. (Giraitis and Phillips). *Suppose* $\{\varepsilon_k, k \geq 0\}$ *are i.i.d. random variables with* $\mathbb{E}\varepsilon_0 = 0$ *and* $\mathbb{E}\varepsilon_0^2 = \sigma^2$. *Under assumptions:* $n(1-\phi_n) \to \infty$, *as* $n \to \infty$ *and* $\mathbb{E}y_0^2 = o(n^{1/2})$ *:*

$$\frac{\left(1-\phi_n^2\right)^{1/2}}{n^{1/2}} \sum_{j=1}^{n} \varepsilon_j y_{j-1} \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0, \sigma^4), \tag{32}$$

$$\frac{1-\phi_n^2}{n} \sum_{j=1}^{n} y_{j-1}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2, \tag{33}$$

$$\frac{(1-\phi_n)}{n^{1/2}} \sum_{j=1}^{n} y_j \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0, \sigma^2). \tag{34}$$

Here $\mathfrak{N}(0,\sigma^2)$ denotes normal distribution with mean zero and variance σ^2.

4.2 Completion of the proof of Theorem 2.1

We explicit here the reduction to the convergence to J of the polygonal line defined by (20). In what follows, we will denote the successive polygonal lines approximating $n^{-3/2}S_n^{\mathrm{pl}}$ by $\pi_{n,i}$ and their vertices by $(l/n, V_{l,i})$. We start with $\pi_{n,1} = n^{-3/2}S_n^{\mathrm{pl}}$ for which

$$V_{l,1} = Y_l = n^{-3/2}\sum_{k=1}^{l} y_{k-1}.$$

We express y_k in terms of innovations

$$y_k = \sum_{j=1}^{k} \mathrm{e}^{(k-j)\gamma/n}\varepsilon_j + e^{k\gamma/n}y_0$$

and for simplicity we shall state that $y_0 = 0$. We can express y_k in terms of W_n^{pl}, noting that $\varepsilon_j = W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) - W_n^{\mathrm{pl}}\left(\frac{j-1}{n}\right)$.

$$\begin{aligned}
y_k &= \sum_{j=1}^{k} \mathrm{e}^{(k-j)\gamma/n}\left(W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) - W_n^{\mathrm{pl}}\left(\frac{j-1}{n}\right)\right)\\
&= \sum_{j=1}^{k} \mathrm{e}^{(k-j)\gamma/n}W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) - \sum_{j=0}^{k-1} \mathrm{e}^{(k-j)\gamma/n}\mathrm{e}^{-\gamma/n}W_n^{\mathrm{pl}}\left(\frac{j}{n}\right)\\
&= W_n^{\mathrm{pl}}\left(\frac{k}{n}\right) + \sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n}(1-\mathrm{e}^{-\gamma/n})W_n^{\mathrm{pl}}\left(\frac{j}{n}\right)\\
&= W_n^{\mathrm{pl}}\left(\frac{k}{n}\right) + \frac{\gamma}{n}\sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n}W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) + \frac{\gamma^2 u_n}{2n^2}\sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n}W_n^{\mathrm{pl}}\left(\frac{j}{n}\right),
\end{aligned}$$

where $u_n \to -1$. Now our first approximation consist in neglecting the last term in the sum above, which gives the polygonal line $\pi_{n,2}$ with

$$V_{l,2} = \frac{1}{n}\sum_{k=1}^{l} W_n\left(\frac{k-1}{n}\right) + \frac{\gamma}{n^2}\sum_{k=1}^{l}\sum_{j=1}^{k-2} \mathrm{e}^{(k-j-1)\gamma/n}W_n\left(\frac{j}{n}\right), \tag{35}$$

where $W_n := n^{-1/2}W_n^{\mathrm{pl}}$ for writing lightness. For the approximation error, we have the bound

$$|V_{l,2} - V_{l,1}| \le \frac{\gamma^2\mathrm{e}^{\gamma}}{2n}\left\|W_n\right\|_\infty .$$

Next, approximating Riemann sums by integrals in (35), we obtain the polygonal line $\pi_{n,3}$ with

$$V_{l,3} = \int_0^{l/n} W_n(s)\,\mathrm{d}s + \frac{\gamma}{n}\sum_{k=1}^{l}\mathrm{e}^{\gamma k/n}\int_0^{k/n}\mathrm{e}^{-\gamma r}W_n(r)\,\mathrm{d}r. \tag{36}$$

Let us estimate the error of approximation. For any $f \in \mathrm{C}[0,1]$, we have

$$\frac{1}{n}\sum_{j=1}^{k-k_0} f\Big(\frac{j+j_0}{n}\Big) - \int_0^{k/n} f(s)\,\mathrm{d}s$$
$$= \sum_{j=1}^{k-k_0} \int_{(j-1)/n}^{j/n} \Big(f\Big(\frac{j+j_0}{n}\Big) - f(s)\Big)\,\mathrm{d}s - \int_{(k-k_0)/n}^{k/n} f(s)\,\mathrm{d}s,$$

whence

$$\left|\frac{1}{n}\sum_{j=1}^{k-k_0} f\Big(\frac{j+j_0}{n}\Big) - \int_0^{k/n} f(s)\,\mathrm{d}s\right| \le \omega\Big(f, \frac{1+j_0}{n}\Big) + \|f\|_\infty \frac{k_0}{n}. \tag{37}$$

Moreover,

$$\text{if } f \in \mathrm{H}^o_\alpha[0,1], \quad \omega(f,\delta) \le \omega_\alpha(f,\delta)\delta^\alpha. \tag{38}$$

If $f(t) = g(t)h(t)$ with g of class C^1 and $h \in \mathrm{C}[0,1]$,

$$\omega(gh,\delta) \le \|g\|_\infty\, \omega(h,\delta) + \|g'\|_\infty \|h\|_\infty\, \delta. \tag{39}$$

Using (37)–(39), we obtain the uniform bound

$$|V_{l,3} - V_{l,2}| \le \frac{1+\gamma \mathrm{e}^\gamma}{n^\alpha}\omega_\alpha\Big(W_n, \frac{1}{n}\Big) + \frac{\gamma \mathrm{e}^\gamma(2+\gamma\mathrm{e}^\gamma)}{n}\|W_n\|_\infty.$$

Finally, we replace the last sum remaining in (36) by an integral of the function $f_n(s) := \mathrm{e}^{\gamma s}\int_0^s \mathrm{e}^{-\gamma r} W_n(r)\,\mathrm{d}r$, noting that $|f_n'(s)| \le (1+\gamma\mathrm{e}^\gamma)\|W_n\|_\infty$. This gives the polygonal line $\pi_{n,4}$ with vertices

$$V_{l,4} = \int_0^{l/n} W_n(s)\,\mathrm{d}s + \gamma\int_0^{l/n} \mathrm{e}^{\gamma s}\int_0^s \mathrm{e}^{-\gamma r} W_n(r)\,\mathrm{d}r. \tag{40}$$

The approximation error is given by the uniform bound

$$|V_{l,4} - V_{l,3}| \le \frac{1+\gamma\mathrm{e}^\gamma}{n}\|W_n\|_\infty.$$

Noting that $\pi_{n,4}$ is exactly the polygonal line defined by(20), gathering all the estimate of errors above, recalling (19), we obtain finally with some positive constants C_γ and C'_γ:

$$\Big\|n^{-3/2}S_n^{\mathrm{pl}} - \pi_{n,4}\Big\|_\alpha \le C_\gamma \omega_\alpha\Big(W_n, \frac{1}{n}\Big) + C'_\gamma\|W_n\|_\infty n^{\alpha-1}. \tag{41}$$

Recalling (16) and (17), it follows that

$$\Big\|n^{-3/2}S_n^{\mathrm{pl}} - \pi_{n,4}\Big\|_\alpha \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

4.3 Maximal inequality

Here we give a detailed proof of Lemma 3.1. It is convenient to start with the following weaker result which already contains the estimate $\max_{1\le k\le n}|y_k| = O_P(n^{1/2}\gamma_n^{-\alpha})$ if $\mathbb{E}\,|\varepsilon_0|^p < \infty$.

Lemma 4.3. *Let $(\eta_j)_{j\ge 0}$ be a sequence of i.i.d. random variables, with $\mathbb{E}\eta_0 = 0$ and $\mathbb{E}\,|\eta_0|^q < \infty$ for some $q \ge 2$. Define*

$$z_k = \sum_{j=1}^{k} \phi_n^{k-j}\eta_j. \tag{42}$$

Then for every $n \ge n_0(q)$, and every $\lambda > 0$,

$$P\left(\max_{1\le k\le n}|z_k| > \lambda\right) \le \frac{4C_q \mathrm{e}^q \mathbb{E}\,|\eta_0|^q}{\lambda^q} n^{q/2}\gamma_n^{1-q/2}, \tag{43}$$

where C_q is the universal constant in the Rosenthal inequality of order q. Choosing $\lambda = n^{1/2}\gamma_n^{1/q-1/2}t$ for arbitrary $t > 0$ provides:

$$\max_{1\le k\le n}|z_k| = O_P\left(n^{1/2}\gamma_n^{1/q-1/2}\right).$$

Proof. The idea of the proof relies on the following observation. For $a < k \le b$,

$$|z_k| = \phi_n^k \left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right| \le \phi_n^a \left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right|.$$

Here $\{\sum_{j=1}^{k}\phi_n^{-j}\eta_j,\ a < k \le b\}$ is a martingale adapted to its natural filtration and if we repeat this procedure with regularly spaced bounds a and b, we keep the structure of a geometric sum for the coefficients ϕ_n^a. To profit of these two features we are lead to the following splitting:

$$n = MK, \quad \max_{1\le k\le n}|z_k| = \max_{1\le m\le M}\ \max_{(m-1)K<k\le mK}|z_k|,$$

where M and K (not necessarily integers) depend on n in a way which will be precised later. Applying this splitting we obtain first:

$$P\left(\max_{1\le k\le n}|z_k| > \lambda\right) \le \sum_{1\le m\le M} P\left(\phi_n^{(m-1)K}\max_{1\le k\le mK}\left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right| > \lambda\right).$$

Then using Markov's and Doob's inequalities at order q gives

$$P\left(\max_{1\le k\le n}|z_k| > \lambda\right) \le \sum_{1\le m\le M}\frac{\phi_n^{q(m-1)K}T_m}{\lambda^q} \quad \text{where} \quad T_m := \mathbb{E}\left|\sum_{1\le k\le mK}\phi_n^{-j}\eta_j\right|^q. \tag{44}$$

To bound T_m we treat separately the special case $q = 2$ with a simple variance computation and use Rosenthal inequality in the case $q > 2$. In both cases, the following elementary estimate is useful.

$$\sum_{1\leq j\leq mK} \phi_n^{-jq} = \frac{\phi_n^{-q[mK]} - 1}{\phi_n^q(\phi_n^{-q} - 1)} \leq \frac{\phi_n^{-q[mK]}}{1 - \phi_n^q} \leq \frac{\phi_n^{-qmK}}{1 - \phi_n},$$

recalling that $0 < \phi_n < 1$, whence,

$$\sum_{1\leq j\leq mK} \phi_n^{-jq} \leq \frac{n}{\gamma_n}\phi_n^{-qmK}. \tag{45}$$

Now in the special case $q = 2$, we have

$$T_m = \operatorname{Var}\left(\sum_{j=1}^{k} \phi_n^{-j}\eta_j\right) = \mathbb{E}\eta_0^2 \sum_{1\leq j\leq mK} \phi_n^{-2j},$$

so by (45),

$$T_m \leq \frac{n}{\gamma_n}\phi_n^{-2mK}\mathbb{E}\eta_0^2. \tag{46}$$

When $q > 2$, we apply Rosenthal inequality which gives here

$$T_m \leq C_q \left(\left(\mathbb{E}\eta_0^2\right)^{q/2} \left(\sum_{1\leq j\leq mK} \phi_n^{-2j} \right)^{q/2} + \mathbb{E}\left|\eta_0\right|^q \sum_{1\leq j\leq mK} \phi_n^{-jq} \right).$$

As $q > 2$, $\left(\mathbb{E}\eta_0^2\right)^{q/2} \leq \mathbb{E}\left|\eta_0\right|^q$. Also we may assume without loss of generality that $\frac{n}{\gamma_n} \geq 1$, so $\frac{n}{\gamma_n} \leq \left(\frac{n}{\gamma_n}\right)^{q/2}$. Then using (45), we obtain

$$T_m \leq 2C_q\mathbb{E}\left|\eta_0\right|^q n^{q/2}\gamma_n^{-q/2}\phi_n^{-qmK}. \tag{47}$$

Note that (46) obtained in the special case $q = 2$ can be included in this formula by defining $C_2 := 1/2$.

Going back to (44) with this estimate, we obtain

$$\begin{aligned} P\left(\max_{1\leq k\leq n} |z_k| > \lambda\right) &\leq 2C_q\mathbb{E}\left|\eta_0\right|^q n^{q/2}\gamma_n^{-q/2}\lambda^{-q} \sum_{1\leq m\leq M} \phi_n^{-Kq} \\ &\leq 2C_q\mathbb{E}\left|\eta_0\right|^q n^{q/2}\gamma_n^{-q/2}\lambda^{-q} M\phi_n^{-Kq}. \end{aligned}$$

Now, choosing $K = \frac{n}{\gamma_n}$, we see that ϕ_n^{-Kq} converges to e^q, so for $n \geq n_0(q)$, $\phi_n^{-Kq} \leq 2\mathrm{e}^q$. Then (43) follows by pluging this upper bound in the inequality above and noting that $M = \gamma_n$. □

Proof of Lemma 3.1. It is convenient to rewrite the assumption (21) as

$$P(|\varepsilon_0| > t) = \frac{f(t)}{t^p}, \quad f(t) \xrightarrow[t\to\infty]{} 0.$$

Moreover

$$f^*(b) := \sup_{t\geq b} f(t) \xrightarrow[b\to\infty]{} 0.$$

In the special case where $p = 2$, (21) is replaced by $\mathbb{E}\varepsilon_0^2 < \infty$, but the above representation of $P(|\varepsilon_0| > t)$ remains valid since $f(t) = t^2 P(|\varepsilon_0| > t) \leq \mathbb{E}(\varepsilon_0^2 \mathbf{1}_{\{|\varepsilon_0|>t\}})$ by Markov inequality and this upper bound goes to zero by dominated convergence theorem.

Let us fix arbitrary positive numbers δ and ϵ, and introduce the truncated random variables

$$\varepsilon'_j = \varepsilon_j \mathbf{1}_{\{|\varepsilon_j|\leq b_n\}} \qquad \widetilde{\varepsilon'_j} = \varepsilon'_j - \mathbb{E}\varepsilon'_j$$
$$\varepsilon''_j = \varepsilon_j \mathbf{1}_{\{|\varepsilon_j|> b_n\}} \qquad \widetilde{\varepsilon''_j} = \varepsilon''_j - \mathbb{E}\varepsilon''_j,$$

where the truncation level b_n will be precised later. Since $\mathbb{E}\varepsilon_j = 0$, $\varepsilon_j = \widetilde{\varepsilon'_j} + \widetilde{\varepsilon''_j}$. Now let us recall that

$$y_k = \phi_n^k y_0 + \sum_{j=1}^{k} \phi_n^{k-j} \varepsilon_j = \phi_n^k y_0 + \widetilde{z'_k} + \widetilde{z''_k},$$

where $\widetilde{z'_k}$ and $\widetilde{z''_k}$ substituting η_j by $\widetilde{\varepsilon'_j}$ and $\widetilde{\varepsilon''_j}$ respectively in the definition of z_k, see(21) above. Then for positive λ, whose dependence on n will be precised later,

$$P\left(\max_{1\leq k\leq n} |y_k| > 3\lambda\right) \leq P_n + P'_n + P''_n, \tag{48}$$

where

$$P_n := P(|y_0| > \lambda), \quad P'_n := P\left(\max_{1\leq k\leq n} \left|\widetilde{z'_k}\right| > \lambda\right), \quad P''_n := P\left(\max_{1\leq k\leq n} \left|\widetilde{z''_k}\right| > \lambda\right).$$

To bound P'_n, applying Lemma 4.3 to $\widetilde{z'_k}$ gives for any $q > p$

$$P'_n \leq \frac{4\mathrm{e}^q C_q \mathbb{E}|\widetilde{\varepsilon'_0}|^q}{\lambda^q} n^{q/2} \gamma_n^{1-q/2} \leq \frac{2^{q+2}\mathrm{e}^q C_q \mathbb{E}|\varepsilon'_0|^q}{\lambda^q} n^{q/2} \gamma_n^{1-q/2},$$

since by elementary convexity inequalities, $\mathbb{E}|\widetilde{\varepsilon_0'}|^q \le 2^q \mathbb{E}|\varepsilon_0'|^q$. Now

$$\begin{aligned}\mathbb{E}|\varepsilon_0'|^q = \int_0^\infty q t^{q-1} P\left(|\varepsilon_0|\,\mathbf{1}_{\{|\varepsilon_j|\le b_n\}} > t\right)\,\mathrm{d}t &= \int_0^{b_n} q t^{q-1} P(t < |\varepsilon_0| \le b_n)\,\mathrm{d}t\\ &\le \int_0^{b_n} q t^{q-1} P(|\varepsilon_0| > t)\,\mathrm{d}t\\ &= \int_0^{b_n} q t^{q-1} \frac{f(t)}{t^p}\,\mathrm{d}t\\ &\le \frac{q\,\|f\|_\infty}{q-p} b_n^{q-p}.\end{aligned}$$

Going back to P_n' we find that

$$P_n' \le \frac{2^{q+2}\mathrm{e}^q q C_q \|f\|_\infty}{q-p} \times \frac{n^{q/2}\gamma_n^{1-q/2} b_n^{q-p}}{\lambda^q}.$$

Now we choose $\lambda = n^{1/2}\gamma_n^{1/p-1/2}\delta$, $q = p+1$ and $b_n = \delta^{p+1}\epsilon\gamma_n^{1/p}$. This gives

$$P_n' = P\left(n^{-1/2}\gamma_n^\alpha \max_{1\le k\le n} \left|\widetilde{z_k'}\right| > \delta\right) \le C_p'\epsilon, \tag{49}$$

with $C_p' = 2^{p+3}\mathrm{e}^{p+1}(p+1)C_{p+1}\|f\|_\infty$.

To bound P_n'', we apply Lemma 4.3 with $z_k = \widetilde{z_k''}$ and $q = 2$ (keeping the above choices of λ and b_n which do not depend on q):

$$P_n'' \le \frac{8\mathrm{e}^2}{\delta^2}\gamma_n^{1-2/p}\mathbb{E}(\varepsilon_0'')^2.$$

In the special case where $p = 2$, this reduces to

$$P_n'' \le \frac{8\mathrm{e}^2}{\delta^2}\mathbb{E}(\varepsilon_0^2\mathbf{1}_{\{|\varepsilon_0|>b_n\}})$$

and this bound goes to zero by Lebesgue's dominated convergence theorem. When $p > 2$, we estimate $\mathbb{E}(\varepsilon_0'')^2$ as follows.

$$\begin{aligned}\mathbb{E}(\varepsilon_0'')^2 &= \int_0^\infty 2tP\left(|\varepsilon_0|\mathbf{1}_{\{|\varepsilon_0|>b_n\}} > t\right)\,\mathrm{d}t\\ &= \int_0^{b_n} 2tP(|\varepsilon_0| > b_n)\,\mathrm{d}t + \int_{b_n}^\infty 2tP(|\varepsilon_0| > t)\,\mathrm{d}t\\ &= b_n^2 P(|\varepsilon_0| > b_n) + \int_{b_n}^\infty 2t^{1-p} f(t)\,\mathrm{d}t\\ &\le f(b_n)b_n^{2-p} + \frac{2}{p-2} f^*(b_n) b_n^{2-p}\\ &\le \frac{p}{p-2}\delta^{(p+1)(2-p)}\epsilon^{2-p}\gamma_n^{2/p-1} f^*(b_n).\end{aligned}$$

Finally, we see that there is a constant $C''_{\delta,\epsilon,p}$ such that for $p \geq 2$,

$$P''_n \leq C''_{\delta,\epsilon,p} f^*(b_n). \tag{50}$$

Going back to (48) with (49) and (50), we obtain

$$Q_n := P\left(n^{-1/2}\gamma_n^\alpha \max_{1\leq k\leq n} |y_k| > \delta\right) \leq P\left(|y_0| > \delta n^{1/2}\gamma_n^{-\alpha}\right) + C'_p\epsilon + C''_{\delta,\epsilon,p} f^*(b_n).$$

This gives $\limsup_{n\to\infty} Q_n \leq C'_p\epsilon$ and as ϵ is arbitrary, (22) follows. □

Remark 4.4. For the sake of simplicity, we assumed throughout the paper that the initialization $y_{n,0} = y_0$ does not depend on n. From the above proof it is obvious that lemma 3.1 remains valid in the more general setting where $y_{n,0}$ depends on n and is $o_P(n^{-1/2}\gamma_n^\alpha)$.

References

[1] P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1986.

[2] W. Cumberland and Z. Sykes. Weak convergence of an autoregressive process used in modelling population growth. *J. Appl. Probab.*, 19:450–455, 1982.

[3] G. Evans and N. Savin. Testing for unit roots: 1. *Econometrica*, 49(3):753–779, 1981.

[4] G. Evans and N. Savin. Testing for unit roots: 2. *Econometrica*, 52(5):1241–1269, 1984.

[5] L. Giraitis and P. Phillips. Uniform limit theory for stationary autoregression. *Journal of time series analysis*, 27(1):51–60, 2006.

[6] D. Hamadouche. Invariance principles in Hölder spaces. *Portugal. Math.*, 57:127–151, 2000.

[7] M. Juodis, A. Račkauskas, and C. Suquet. Hölderian functional central limit theorems for linear processes. *Alea : Latin American journal of probability and mathematical statistics*, 5:47–64, 2009.

[8] P. Phillips. Towards a unified asymptotic theory for autoregression. *Biometrika*, 74(3):535–547, 1987.

[9] A. Račkauskas and C. Suquet. Necessary and sufficient condition for the functional central limit theorem in Hölder spaces. *Journal of Theoretical Probability*, 17(1):221–243, 2004.

[10] C. Suquet. Tightness in Schauder decomposable Banach spaces. In *Amer. Math. Soc. Transl.*, volume 193, pages 201–224. AMS, 1999.