PUB. IRMA, LILLE 2011
Vol. 71, N° VI

Asymptotic study of a busy period in a retrial queue *

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Résumé

Dans ce papier, on propose deux approches pour étudier le comportement asymptotique d'une période d'activité d'une file d'attente M/G/1. La première approche est basée sur la modélisatuion d'Artaléjo et Falin (1996) et le principe d'invariance pour des variables aléatoires indépendantes. Dans la seconde approche, on utilise la modélisation de l'évolution du système en termes de périodes d'activité et de périodes d'inactivité du serveur et on conclut également avec un principe d'invariance hölderien.

Abstract

In this work, we propose two approaches to study the convergence in distribution of the busy period of the M/G/1 retrial queue. The first approach rely on the modeling of Artaléjo and Falin (1996) and an invariance principle for independent random variables. In the second one, we use the evolution of the system in terms of idle periods and busy periods of the server and we conclude too with an Hölderian invariance principle.

Keywords : Brownian motion, Hölder space, invariance principle, retrial queue, Schauder decomposition, tightness.
AMS classifications : 60B10, 60F05, 62G30, 60K25.

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*Preprint. This work was partially done during the visit of the third author in the Laboratoire P. Painlevé of the University Lille I (France).

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In practice, retrial queueing systems are very used in telecommunication network modelling and the modelling of computer systems. A review of main results and description of situations where retrial queues arise can be found in the didactical books of Falin and Templeton (1997) and Artalejo and Gomez-Corall (2008). The reader can also refer to the works of Falin and Templeton (1997), Yang and Templeton (1987) and Falin (1990).

In the analysis of these systems, one is generally interested to the analysis of the performance characteristics as the waiting time of a customer in the system, the busy period of the system, the number of customers served during the busy period, ... Our interest, in this work is limited to the busy period of the M/G/1 retrial queue. The analysis of this characteristic is very important from the server's point of view and is also useful in the efficient organization of the system resources.

The structure of the busy period in the M/G/1 retrial queue and its analysis with Laplace-Stieltjes transform were studied by several methods (see Falin Templeton and (1997) and Falin (1979), Artalejo (1999)). Among the approaches used to get an estimation for the density function of a busy period, one can mention the principle of maximum entropy and the approach based on a truncated retrial queue (Artalejo and Gomez-Corall (2008)).

We describe the $M/G/1$ retrial queue as follows. Primary

scustomers arrive according to Poisson flow of rate $\lambda > 0$. Any customer who finds the server busy must leave the service area in order to seek service again at subsequent epochs until he finds the server free. Between retrials a customer is said in orbit. Delays between retrials of each customer in orbit are i.i.d. exponentially distributed with rate μ. A customer (primary arrival or retrial) who finds the server free is served immediately. The sequence of service times is a sequence of i.i.d. random variables with distribution B on R_+. Let $\beta(s)$, $\beta_k = (-1)^k\beta^{(k)}(0)$, for $k \in \mathbb{N}$, be, respectively, the Laplace transform and the moments of the service time. We assume that $B(0+) = 0$, that is service time can not be zero. Interarrival times, service times, and retrial times are mutually independent random variables.
Let η_i be the epoch at which the i^{th} service completion occurs, ξ_i the epoch at which the i^{th} service starts, and π_i the arrival epoch of the i^{th} primary customer. We also denote $\rho = \lambda\beta_1$.
The system evolves in the following way. At epoch η_{i-1}, the $(i-1)^{th}$ customer completes its service and the server becomes free. The next customer enters service after some random time R_i, during which the server is free. If the number of customers in orbit at time η_{i-1}, N_{i-1} is equal to n, then R_i is exponentially distributed with rate $\lambda + n\mu$. The i^{th} service time corresponds to a primary customer with probability $\frac{\lambda}{\lambda+n\mu}$ and it corresponds to a repeated attempt with probability $\frac{n\mu}{\lambda+n\mu}$. The i^{th} service time S_i start at epoch $\xi_i = \eta_{i-1} + R_i$. Note that repeated attempts that occur during this service time do not modify the state of the system . At epochs $\eta_i = \xi_i + S_i$, the server becomes idle again. Thus, the evolution of our retrial queue is described in terms of an alternating sequences (R_i, S_i).

Our goal in this work is the study of the busy period using invariance principles.
- The first approach, rely on the result given by Artalejo and Falin (1996). In their work, they show an expression of the busy period of the M/G/1 retrial queue in terms of the orbit idle period $L^{(i)}$ and the orbit busy period $L^{(b)}$. Using these two notions and an invariance principle, we propose an asymptotic study of the busy period of the system.
- In the second one, we use the evolution of the systeme according to the sequence $(R_i,\ S_i)$ of busy periods and idle periods of the server. A busy period of the system is then expressed in terms of several service times S_i and a time sold between two service R_i, whereas there is customers in the orbit. So, a busy period L is the sum of n dependent random variables, given that n is the number of customer served during this period. We start by estimating the number of customer in the orbit during L and we use an invariance principle to study the sum obtained.

1 Orbit idle period and orbit busy periods

The evolution of the system is described in terms of an alternating sequences $(R_i,\ S_i)$ of idle and busy period of the server. The main characteristics of this sequence is the dependence of R_i on N_{i-1}. In order to overcome this difficulty, Artalejo and Falins (1996) introduced the orbit busy period ($L^{(b)}$) and the orbit idle period ($L^{(i)}$) which are defined as follow.

- The orbit idle period $L^{(i)}$ is the period that starts at an epoch when a customer alone in the orbit produces a repeated call and finds the server idle (thus the orbit becomes empty), and ends when a primary customer finds the server busy and is obliged to join the orbit.

- The orbit busy period $L^{(b)}$ is the period that starts at an epoch when a primary customer arrives and finds the server busy and the orbit idle, and ends at the next epoch at which a repeated attempt finds the server idle and the orbit becomes empty.

The ordinary busy period is defined as the period starting at an epoch t_0 with the arrival of a customer who finds the system empty and ends at the first service completion epoch t_1 at which the system becomes empty again.

In the following, we present a description of this period based on the sequence $(L_k^{(i)}, L_k^{(b)})$ of alternating idle and busy periods of the orbit .

Let T_k be the random duration of the competition between a service time and arrival input which takes place just before the beginning of $L_k^{(b)}$. Thus T_k ends with the arrival of a primary customer.

Figure 1 shows the evolution of the system.

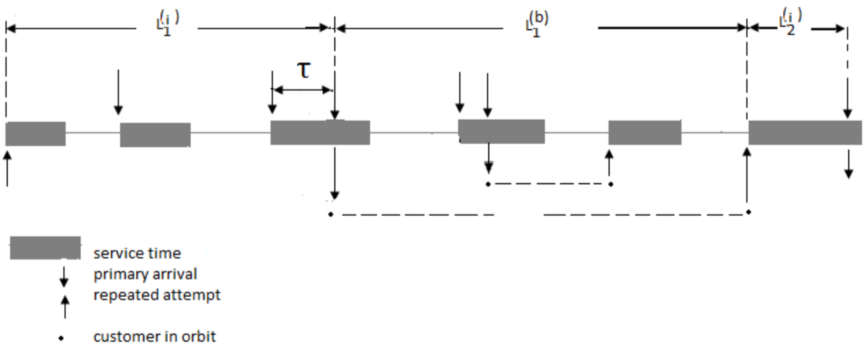

Figure 1: The system behavior

The duration of $L_k^{(i)}$ is determined by the minimum between the service time and the arrival of a new customer. Using conditional probabilities, we get that (see Artalejo and Falin (1996))[1]:

- the conditional density function of the duration of this competition, that it ended in completion of the service time is

$$f_1(t) = \frac{1}{\beta(\lambda)} B'(t) e^{-(\lambda t)}.$$

- the conditional density function of the duration of this competition, that it ended in the arrival of a customer is

$$f_2(t) = \frac{1}{1-\beta(\lambda)} \lambda e^{-(\lambda t)} (1 - B(t)).$$

According to the work of Artalejo and Falin (1996), a busy period L is expressed in terms of several orbit busy periods L_k^b and periods of competition between the service time and the poisson input process (the arrival of a primary customer) T_k. Ω is the competition that leads to the end of L, so Ω ends in completion of the service.
The distribution of $L_k^{(b)}$ depend on the distribution of T_k. consequently, $L_k^{(b)}$ is not a sequence of i.i.d. random variables. However, if we consider the sequence $C_k^b = T_k + L_k^{(b)}$, then we can decompose L in terms of i.i.d. random variables

(see Artalejo and Falin (1996)).
Let L be a busy period of the system. We also denote $N^{(b)}$ the number of orbit busy period which take place during the busy period L. Then, we can write:

$$L_n = (L/_{N^{(b)}=n-1}) = (\sum_{k=1}^{k=n-1} C_k^b) + \Omega,\ n \in \mathbb{N}^* - \{1\}. \tag{1.1}$$

Figure 2 given by Artalejo and Falin 1996, illustrates the decomposition of a busy period L in terms of the random variable C_k^b. In the graph, τ_1, τ_2 are realization of the variables T_1, T_2 respectively and ω is a realization of Ω.

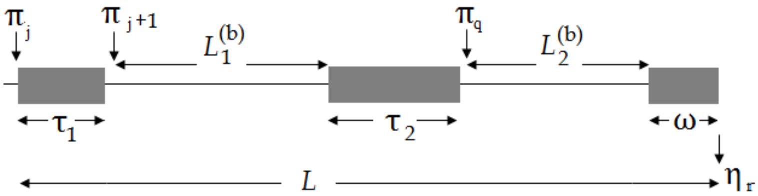

Figure 2: A busy period of the system: L, where $t_0 = \pi_j$ and $t_1 = \eta_r$

In this example, we see that

$$(L/_{N^{(b)}=2}) = T_1 + L_1^b + T_2 + L_2^b + \Omega$$

$$(L/_{N^{(b)}=2}) = (\sum_{k=1}^{k=2} C_k^b) + \Omega.$$

Remark 1.1. - *The density function of* Ω *is* f_1 *[1].*
- *All the random variables* $T_k,\ \ k \geq 1$ *have a same distribution* T*. The density of* T *is* $f_2(t)$ *[1].*
- *The random variables* $L_k^b,\ k \geq 1$ *are of the same distribution* $L^{(b)}$*. Consequently, the random variables* $C_k^b,\ \ k \geq 1$ *are identically distributed. Henceforth it is denoted by* C^b*.*

Using the notations:

$X_k = C_{k-1}^b,\ \ k \geq 2$ and $X_1 = \Omega$, we can write $L_n = \sum_{k=1}^{k=n} X_k$.

In the following, we shall use the notations: for $n \geq 1$,

$$a_n = E(\Omega) + (n-1)E(C^b) = -\frac{\beta'(\lambda)}{\beta(\lambda)} + (n-1)\frac{\beta(\lambda)p_{00}^{-1} + \lambda\beta'(\lambda) - \beta(\lambda)}{\lambda(1-\beta(\lambda))},$$

$$s_n^2 = Var(\Omega) + (n-1)Var(C^b),$$

with $Var(\Omega)$),with $Var(C^b)$ are obtained from the formulas (2.2), (2.3) and (2.4).

Remark 1.2. *Under the condition $\rho < 1$, it is shown in (Artalejo et Falin 1996) that*

$$E(L^b) = \frac{\beta(\lambda)p_{00}^{-1} - 1}{\lambda(1-\lambda)} \quad et \quad E(T) = \frac{1}{\lambda} + \frac{\beta'(\lambda)}{1-\beta(\lambda)}$$

Consequently,

$$E(C^b) = \frac{\beta(\lambda)p_{00}^{-1} + \lambda\beta'(\lambda) - \beta(\lambda)}{\lambda(1-\beta(\lambda))} \tag{1.2}$$

where $p_{00} = (1-\rho)\exp(\frac{-\lambda}{\mu}\int_0^1 \frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}du)$.

We need the following Lemma.

Lemma 1.1. *If $\rho < 1$, then*

$$E(\Omega) = -\frac{\beta'(\lambda)}{\beta(\lambda)}, \qquad E(\Omega^2) = \frac{\beta''(\lambda)}{\beta(\lambda)}. \tag{1.3}$$

$$E((C^b)^2) = \frac{\lambda^2\beta''(\lambda) - 2(p_{00}^{-1}-1)\lambda\beta'(\lambda) + \beta(\lambda)(-E(L^2) + 2(p_{00}^{-1}-1)^2)}{\lambda^2(1-\beta(\lambda))}. \tag{1.4}$$

where $E(L^2)$ is given in Artalejo and Lopez-Herrero (2000):

$$E(L^2) = \frac{1}{p_{00}}\left[\frac{1}{1-\rho^2}\left(\frac{2\rho\beta_1}{\mu} + \beta_2\right) - \int_0^1 \frac{2}{\lambda\mu(\beta(\lambda-\lambda t)-t)}\times \right.$$
$$\left.\left(1 - \frac{\lambda(1-t)\beta'(\lambda-\lambda t)}{\beta(\lambda-\lambda t)-t} - \frac{1}{1-\rho}\exp\{\frac{\lambda}{\mu}\int_t^1 \frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}du\}\right)dt\right]$$

Proof of the Lemma 1.1
1) The Laplace transform of Ω is given by

$$\Psi(s) = \frac{\beta(s+\lambda)}{\beta(\lambda)}$$

We derive this functionwith respect to s, we obtain

$$\Psi'(s) = \frac{\beta'(s+\lambda)}{\beta(\lambda)}$$

we know that $\Psi(0) = -E(\Omega)$, then

$$E(\Omega) = -\frac{\beta'(\lambda)}{\beta(\lambda)}$$

In order to obtain $E(\Omega^2)$, it is sufficient to derive this function a second time with respect to s and to take $s = 0$, with $\Psi''(0) = E(\Omega^2)$.

2) Let $g(s)$ and $\Phi(s)$ be the Laplace transforms of C^b and L respectively. We have the following relation (Artalejo and Falin, 1996)

$$g(s) = \frac{1}{1-\beta(\lambda)}(1 - \frac{\beta(s+\lambda)}{\Phi(s)}).$$

The first derivative of g with respect to s give

$$g'(s) = \frac{-1}{1-\beta(\lambda)}(\frac{\beta'(s+\lambda)\Phi(s) - \Phi'(s)\beta(s+\lambda)}{(\Phi(s))^2}).$$

We compute the second derivative :

$$g''(s) = \frac{-1}{1-\beta(\lambda)}\frac{1}{\Phi^4(s)}\left[\beta''(s+\lambda)\Phi(s) + \beta(s+\lambda)(-\Phi''(s) + 2(\Phi'(s))^2\phi(s)) \right.$$
$$\left. -2\phi'(s)(\phi(s))^2\beta'(s+\lambda)\right]$$

It is known that $E((C^b)^2) = g"(0)$, consequently

$$E((C^b)^2) = \frac{1}{\lambda^2(1-\beta(\lambda))}[\lambda^2\beta"(\lambda) - 2(p_{00}^{-1}-1)\lambda\beta'(\lambda) + \beta(\lambda)(-E(L^2) + 2(p_{00}^{-1}-1))^2].$$

Corollary 1.1. *For $\rho < 1$, we suppose that the random variables Ω et C^b are non degenerate and there exist $\gamma > 2$ such that $\forall k \in \mathbb{N}^*, E|(X_k - E(X_k))|^\gamma < \infty$. Then, the sequence*

$$\frac{1}{s_n}\left(\sum_{k=1}^{[nt]}\big(X_k - E(X_k)\big) + (nt - [nt])\big(X_{[nt]+1} - E(X_{[nt]+1})\big)\right), \quad t \in [0,1]$$

converges weakly to the standard Brownian motionvers in $H_\alpha[0,1]$ for all $0 < \alpha < \frac{1}{2} - \frac{1}{\gamma}$. In addition, we have the convergence in distribution in $\mathbb{R}$:

$$\frac{L_n - a_n}{s_n} \longrightarrow N(0,1)$$

where $N(0,1)$ is a standard normal random variable.

Before demonstrating the Corollary 2.1, let us recall the theorem of Hamadouche and Taleb (2009). We define the Hölder space $H_\alpha[0,1]$, α, $(0 < \alpha \leq 1)$, as the space of functions f defined on $[0,1]$, vanishing at zero such that

$$\|f\|_\alpha = \sup_{0<|t-s|\leq 1}\frac{|f(t)-f(s)|}{|t-s|^\alpha} < +\infty.$$

Théorème 1.1. *(Hamadouche and Taleb)*
Let $(Y_n)_{n\geq 1}$ be a sequence of centered and independent random variables, non identically distributed. We assume there exist $\gamma > 2$, two positives constants m

and M such that $\forall j \in \mathbb{N}^*$,

$$m \leq E(Y_j^2) \quad et \quad E|Y_j|^\gamma \leq M < \infty$$

Then for $n \in \mathbb{N}^*$, the sequence of the smoothed partial sum process

$$\zeta_n(t) = \frac{1}{b_n}\left(\sum_{i=1}^{[nt]} Y_i + (nt - [nt])Y_{[nt]+1}\right), \quad t \in [0,1],$$

converges weakly to standard Brownian motion W in H_α, for all $0 < \alpha < \frac{1}{2} - \frac{1}{\gamma}$. Where $b_n^2 = \sum_{j=1}^n Var(Y_j)$.

Proof of Corollary 2.1.
Step 1 We consider the sequence $(Y_k = X_k - E(X_k)$, $k \geq 1$, and we shall verify that conditions of Hamadouche and Taleb theorem are satisfied. Indeed, the sequence $\zeta_n \in H_\alpha[0,1]$ and
1) To satisfy the condition $m \leq E(Y_j^2)$, we have
• $E(Y_1^2) = Var(X_1) = Var(\Omega)$.
• for $k \geq 2$, $EY_k^2 = Var(X_k) = Var(C^b)$.

The variables Ω and C^b are non degenerate then, $\min(Var(\Omega^2),\ Var(C^b)) > 0$. Then, it is sufficient to take $m = \min(Var(\Omega^2),\ Var(C^b))$.
2) To satisfy the condition $E|Y_j|^\gamma \leq M < \infty$, it is sufficient to take $M = \max(E|\Omega - E(\Omega)|^\gamma,\ E|C^b - E(C^b)|^\gamma)$, because
• $Y_1 = \Omega - E(\Omega)$;
• For $k \geq 2$, $Y_k = C^b - E(C^b)$.
3) Let's calculate b_n^2

$$b_n^2 = \sum_{k=1}^{k=n} Var(Y_k)$$

Consequently

$$b_n^2 = \sum_{k=1}^{k=n} E(Y_k^2) = \sum_{k=1}^{k=n} Var(X_k) = Var(\Omega) + \sum_{k=2}^{k=n} Var(C^b).$$

It follows that

$$b_n^2 = Var(\Omega) + (n-1)Var(C^b)$$

which means that

$$b_n^2 = s_n^2.$$

Using Hamadouche and Taleb theorem, we deduce that the sequence

$$\frac{1}{s_n}\sum_{k=1}^{[nt]}\big(X_k - E(X_k)\big) + \frac{1}{s_n}(nt-[nt])\big(X_{[nt]+1} - E(X_{[nt]+1})\big), \quad t \in [0,1],\ n \in \mathbb{N}^*$$

converges to the standard Brownian motion in H_α for all $0 < \alpha < \frac{1}{2} - \frac{1}{\gamma}$.
Step 2. Let us consider the continuous function on H_α $F : H_\alpha \longrightarrow \mathbb{R}$, given by $F(g) = g(1)$. Using the continuous mapping theorem function, we get $F(\xi_n)$ converges in distribution in $\mathbb{R}$ to $F(W_t) = W_1$. With $F(W_t) = W_1 = N(0, 1)$ and

$$F(\xi_n) = \frac{\sum_{k=1}^{k=n}(X_i - E(X_i))}{s_n} = \frac{(\sum_{k=1}^{k=n} X_k) - \sum_{k=1}^{k=n} E(X_k)}{s_n}$$

$$= \frac{(\sum_{k=1}^{k=n} X_k) - E(\Omega) - (n-1)E(C^b)}{s_n} = \frac{(\sum_{k=1}^{k=n} X_k) - a_n}{s_n},$$

we obtain the convergence of Corollary 2.1.

2 Approach using the sequence of random variables (R_i, S_i)

As it has been mentioned previously, the system evolves according to the sequence $\{(R_i;\ S_i), i \geq 0\}$ of the busy and idle periods of the server. Thus, a busy period of the system consists of an alternation of a service time and periods where the server is free whereas there are some customers in orbit.

Let L be a busy period of the M/G/1 retrial queue. Let us denote by I the number of customers served during this period, we can write

$$L = \sum_{i=1}^{i=I}(R_i + S_i), \qquad R_1 = 0 \quad .$$

We define then

$$T_n = (L/_{I=n}) = \sum_{i=1}^{i=n}(R_i + S_i), \qquad R_1 = 0 \quad , \quad n \geq 1$$

We set $U_i = R_i + S_i$, for $i \geq 1$. And, we write

$$T_n = (L/_{I=n}) = \sum_{i=1}^{i=n}(U_i)$$

Remark 2.1. *During the random time R_i which take place during a busy period L, there is always customers in the orbit (the orbit is not empty).*

The random variable R_i depends on the history of the system until η_{i-1} only through the number of customer in orbit at this time, N_{i-1}, and has the conditional distribution: $G(x) = P(R_i < x/N_{i-1} = k) = 1 - e^{(-\lambda+k\mu)x}$ with mean $E(R_i/_{N_{i-1}=k}) = \frac{1}{\lambda+k\mu}$.

We recall the following lemma due to Falin.

Lemma 2.1. *(Falin, Templeton (1997))*
Random variables $(S_i + R_i)_{i\geq 2}$ are identically distributed iff random variables N_i are identically distributed.

In the following, we assume that $\rho < 1$ and the number of customer in orbit during the period L is estimated (fixed) by $[\overline{n}_o]+1$, where $\overline{n}_o$ is the mean number of customers in the orbit $\overline{n}_o = \frac{\lambda^2\beta_2}{2(1-\rho)} + \frac{\lambda\rho}{\mu(1-\rho)}$ given in (Yang and Templeton (1987)). The embedded Markov chain $(N_i)_{i\geq 1}$ is then identically distributed. Taking into account Lemma 3.1, the random variables $S_i + R_i$ are identically distributed. Then, we obtain $E(U_1) = \beta_1$ and for $i \geq 1$, $E(U_i) = \frac{1}{\lambda+([\overline{n}_o]+1)\mu}$. Using the notation $(Z_i = U_i - E(U_i))$, we have a sequence (Z_i) of dependent centered random variables and not identically distributed. It is difficult to determine the type of the dependence between these variables, nevertheless if the sequence (Z_i) is strong mixing, we can write the result.

Corollary 2.1. *If $\rho < 1$ and $(Z_n)_{n\geq 1}$ is a sequence of $\alpha-$mixing, strictly stationary random variables. we assume that there are $\gamma > 2$ and $\varepsilon > 0$ such that $E|U_{1i} - E(U_i)|^{\gamma+\varepsilon} < \infty$ and $\Sigma_{n=1}^{\infty}(n+1)^{\frac{\gamma}{2}-1}[\alpha_n]^{\frac{\varepsilon}{\gamma+\varepsilon}} < \infty$. Then for $n \in \mathbb{N}^*$, the sequence*

$$\xi_n(t) = \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{[nt]}(U_i - E(U_i)) + (nt - [nt])(U_{([nt]+1} - E(U_{[nt]+1}))\right) \quad t \in [0,1],$$

converges weakly to the standard Brownian motion W in H_α for all $0 < \alpha < \frac{1}{2} - \frac{1}{\gamma}$. In addition, we have the convergence in distribution in $\mathbb{R}$:

$$\frac{T_n - b_n}{\sigma\sqrt{n}} \longrightarrow N(0,1)$$

with $b_n = \sum_{i=1}^{i=n} E(U_i)$ and $\sigma = Var(U_1) + 2\sum_{i=2}^{i=\infty} Cov(U_1, U_i)$.

Proof of Corollary 3.1
Using the following theorem

Théorème 2.1. *(Hamadouche,(2000)) [9]*
Let $(y_n)_{n\geq 1}$ be a strictly stationary sequence of $\alpha-$mixing and centered random variables. We suppose that there are $\gamma > 2$ et $\varepsilon > 0$ such that $E|Y_1|^{\gamma+\varepsilon} < \infty$

and $\Sigma_{n=1}^{\infty}(n+1)^{\frac{\gamma}{2}-1}[\alpha_n]^{\frac{\varepsilon}{\gamma+\varepsilon}} < \infty$. *Then for* $n \in \mathbb{N}^*$, *the sequence of smoothed partial sums process*

$$\xi_n(t) = \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{[nt]} Y_i + (nt-[nt])Y_{[nt]+1}\right) \qquad t \in [0,1]$$

converges to the standard Brownian motion W *in* H_α^0 *for all* $0 < \alpha < \frac{1}{2} - \frac{1}{\gamma}$. *Where* $\sigma^2 = EY_1^2 + 2\sum_{j=2}^{\infty} Cov(Y_1, Y_j) < \infty$.

We show that under conditions of Corollary 3.1, as in section 2, the convergence in distribution to W of the sequence

$$\zeta_n = \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{[nt]} Z_i + (nt-[nt])Z_{[nt]+1}\right)$$

in $H_\alpha[0,1]$.
Then, we consider the function on $(H_\alpha, \|.\|_\alpha)$; $F : H_\alpha \longrightarrow \mathbb{R}$ given by $F(g) = g(1)$. Using the continuous mapping theorem, we get $F(\xi_n)$ converges in distribution in $\mathbb{R}$, to $F(W_t) = W_1$. After some calculations, we obtain $F(\zeta_n) = \frac{T_n - d_n}{\sigma\sqrt{n}}$. Which concludes the proof of corollary 3.1.

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