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Testing epidemic change in the variance *

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Résumé

Dans ce papier, on propose des statistiques de type DI basées sur des variables aléatoires indépendantes non de même loi ou sur des variables aléatoires dépendantes (mélangeantes). On étudie leurs comportements asymptotiques sous l'hypothèse nulle et on donne pour chaque cas une application pour des tests de rupture de variances.

Abstract

In this paper, we propose statistics of type DI based on independent not identically distributed or $\alpha-$mixing random variables. We obtain their limit distributions under the null hypothesis and we present an application for testing epidemic change in the variance for each case.

Keywords : Brownian bridge, Hölder space, perturbed empirical process, Schauder decomposition, tightness.

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1 Introduction

Let $(X_1, X_2, ..., X_n)$ a sample of random variables with means $m_1, m_2, ..., m_n$ respectively.
We want to test the standard null hypothesis :
$(H_0) : m_1 = m_2 = ... = m_n,$
against the epidemic altrenative:
$(H_A) : \exists 1 < k^* < m^* < n$ such that
$m_1 = ... = m_{k^*} = m_{m^*+1} = ... = m_n, m_{k^*+1} = ... = m_{m^*}$ and $m_{k^*} \neq m_{k^*+1}$.
Writing $l^* = m^* - k^*$ the lenght of the epidemic, we assume that both l^* and $n - l^*$ go to infinity with n.
Set

$$S(0) = 0,\ S(t) = \sum_{k \le t} X_k,\ 0 < t \le n \quad \text{and} \quad t_k = \frac{k}{n}, 0 \le k \le n.$$

Define the polygonal lines $\xi_n = \{\xi_n(t), t \in [0,1]\}$ with vertices $(\frac{k}{n}, S_k)$, $0 \le k \le n$.

$$\xi_n(t) = \sum_{k=1}^{j} X_k + (nt - j)X_{j+1},\ \frac{j}{n} \le t \le \frac{j+1}{n}. \tag{1.1}$$

For a i.i.d. sequence of random variables, classical Donsker-Prokhorov invariance principle states that, if the variance σ^2 is finit, $\sigma^{-1}n^{-\frac{1}{2}}\xi_n$ converges in distribution in $C[0,1]$ to the Brownian motion W.

By continuous mapping, $g(\sigma^{-1}n^{-\frac{1}{2}}\xi_n)$ converges in distribution to $g(W)$, where g is a continuous functional. Several applications are obtained using this property. One classical example is the change point detection of the mean of a sample.

Levine and Kline [6] proposed the test statistic

$$Q = \max_{1 \le i < j \le n} |S(j) - S(i) - S(n)(\frac{j}{n} - \frac{i}{n})|. \tag{1.2}$$

For $g(x) = \sup_{0 \le t \le 1} |x(t)|$, we have $Q = g(\xi_n)$.
Since g is continuous on $C[0,1]$, under the null hypothesis for a i.i.d. sample with variance 1 and fixed mean, we have

$$n^{-\frac{1}{2}} Q \xrightarrow{D} \sup_{0<t<1} |B(t)|,$$

where $B(t) = W(t) - tW(1)$ is the Brownian bridge associated to W.

Under the alternative hypothesis, we can use the statistic proposed by Račkauskas and Suquet

$$UI(n,\alpha) = \max_{1\le i<j\le n} \frac{|S(j)-S(i)-S(n)(\frac{j-i}{n})|}{(\frac{j-i}{n})^\alpha}. \tag{1.3}$$

The associated functional is

$$h(x) = \sup_{0<|t-s|<1} \frac{|x(t)-x(s)|}{|t-s|^\alpha}.$$

h is not continuous on $C[0,1]$, it is continuous on $(H_\alpha^0, \|.\|_\alpha)$.

Lamperti [5] proved that if $0<\alpha<\frac{1}{2}$ and $\mathbb{E}|X_1|^p<\infty$, where $p = (\frac{1}{2}-\alpha)^{-1}$, $n^{-\frac{1}{2}}\sigma^{-1}\xi_n$ converges in distribution to W in H_α^0. This allows us to obtain limit of $h(n^{-\frac{1}{2}}\sigma^{-1}\xi_n)$. This result was generalized by

- Erickson [2] : partial sums processes indexed by $[0,1]^d$.
- Hamadouche [3] : weakly dependent sequences.
- Račkauskas and Suquet [7] : Hölder space H_ρ^0, where $\rho(h) = h^\alpha \ln^\beta(\frac{1}{h})$.
  Let us denote by D_j the set of dyadic numbers in $[0,1]$ of level j :

$$D_0 = \{0,1\}, \quad D_j = \{(2l-1)2^{-j}, \quad 1\le l\le 2^{j-1}\}, \quad j\ge 1.$$

We write $D = \underset{j\ge 0}{\cup} D_j$ et $D^\alpha = D \setminus \{0\}$.
Put for $r\in D_j$, $j\ge 0$,

$$r^- = r-2^{-j} \quad \text{and} \quad r^+ = r+2^{-j}.$$

For any function $x:[0,1]\longrightarrow \mathbb{R}$, define its Schauder coefficients $\lambda_r(x)$ by

$$\lambda_r(x) = x(r) - \frac{x(r^+)+x(r^-)}{2}, \quad r\in D_j, \quad j\ge 1,$$

$\lambda_0(x) = x(0)$ and $\lambda_1(x) = x(1)$.

Dyadic increments statistics are defined by

$$DI(n,\alpha) = \max_{1\le 2^{j\alpha}\le n} \frac{1}{2^{-j\alpha}} \max_{r\in D_j} \left| S(nr) - \frac{1}{2}S(nr^+) - \frac{1}{2}S(nr^-) \right|. \tag{1.4}$$

Let $W = \{W(t), t\in[0,1]\}$ be a standard Wiener process, we define the following random variable:

$$DI(\alpha) = \sup_{j\ge 1} \frac{1}{2^{-j\alpha}} \max_{r\in D_j} \left| W(r) - \frac{1}{2}W(r^+) - \frac{1}{2}W(r^-) \right|. \tag{1.5}$$

One important feature of the Hölderian statistics is the detection of short epidemics. The test statistic Q detects only epidemics whose length l^* is at least of order $n^{\frac{1}{2}}$, while $UI(n, \alpha)$ detects epidemics with length l^* of the order of $n^\delta, 0 < \delta < \frac{1}{2}$.
$UI(n, \alpha)$ and $DI(n, \alpha)$ have the same asymptotical behavior. The statistics $DI(n, \alpha)$ are of special interest because their limiting distribution is explicitly computable.

Our objectif in this paper is to propose statistics of type DI based on independent not identically distributed random variables, in section 2, and on $\alpha-$mixing random variables in section 3. We obtain their limit distributions under the null hypothesis and we present application for testing epidemic change in the variance for each case .

2 Independent non stationary case

2.1 Convergence of $DI(n, \alpha)$

Let $(X_i)_{i\geq 1}$ a sequence of random variables with variances $(\sigma_i^2)_{i\geq 1}$, let $s_n = \sum_{i=1}^{n} \sigma_i^2$.
Under the hypothesis

(H_0') : The random variables X_i are independent with mean 0. We have the result

Theorem 1 Under (H_0), suppose that there exist $\gamma > 2$ and $m > 0$ and $M > 0$ such that

$$m \leq E|X_j|^2 \text{ and } E|X_j|^\gamma \leq M < +\infty, \forall j > 1.$$

Assume that

$$\forall A > 0, \lim_{n\longrightarrow +\infty} \sum_{i=1}^{n} P(|X_i| > A\frac{s_n}{n^{\frac{1}{2}}}) = 0, \forall i \geq 1. \tag{2.1}$$

Then for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$,

$$s_n^{-1} DI(n, \alpha) \xrightarrow[n\longrightarrow +\infty]{\mathcal{D}} DI(\alpha).$$

Proof

First we observe that

$$DI(n, \alpha) = \max_{1\leq 2^{j\alpha}\leq n} \frac{1}{2^{-j\alpha}} \max_{r\in D_j} |\lambda_r(S_{n.})| , \tag{2.2}$$

where $(S_{n.})$ is the discontinuous process $(S(nt), 0 \leq t \leq 1)$ defined by

$$S(nt) = \sum_{k=1}^{[nt]} X_k.$$

This process may be written as

$$S(nt) = \xi_n(t) - (nt - [nt])X_{[nt]+1},$$

from which we see that

$$|\lambda_r(S_{n.}) - \lambda_r(\xi_n)| \leq 2 \max_{1\leq i\leq n} |X_i|\,.$$

Using this estimation in (7), we obtain

(2.3) $$s_n^{-1} DI(n,\alpha) = g_n(s_n^{-1}\xi_n) + Z_n,$$

where $g_n(x) = \max_{1\leq 2^{j\alpha}\leq n} \max_{r\in D_j} \frac{|\lambda_r(x)|}{2^{-j\alpha}}$, $x \in H_\alpha^0$, and the random variable Z_n satisfies

(2.4) $$|Z_n| \leq \frac{2}{\frac{s_n}{n^\alpha}} \max_{1\leq i\leq n} |X_i|.$$

$$P(\max_{1\leq i\leq n} |X_i| > A\frac{s_n}{n^\alpha}) \leq \sum_{i=1}^{n} P(|X_i| > A\frac{s_n}{n^\alpha}).$$

We have

$$P(|X_i| > A\frac{s_n}{n^\alpha}) \leq P(|X_i| > A\frac{s_n}{n^{\frac{1}{2}}}), \quad \forall \alpha < \frac{1}{2}, \forall i \geq 1.$$

Using condition (6), we have the convergence of $\frac{n^\alpha}{s_n} \max_{1\leq i\leq n} |X_i|$ in probability to 0.

The following results are useful to simpify the proof of our theorem
Lemma 1 (Račkauskas, Suquet [8]) Let $(\eta_n)_{n\geq 1}$ be a tight sequence of random elements in the separable Banach space B and g_n, g be continuous functionals $B \to \mathbb{R}$. Assume that g_n converges pointwise to g on B and $(g_n)_n$ is equicontinuous. Then

$$g_n(\eta_n) = g(\eta_n) + o_P(1).$$

Lemma 2 (Račkauskas, Suquet [8]) Let $(B, \|\ .\ \|)$ a vector normed space and $q : B \longrightarrow \mathbb{R}$ such that

(a) q is subadditive : $q(x+y) \leq q(x) + q(y), x, y \in B$.
(b) q is symetric : $q(-x) = q(x), x \in B$.
(c) For some constant $C, q(x) \leq C \parallel x \parallel, x \in B$.
Then q satisfies the Lipschitz condition

$$|q(x+y) - q(x)| \leq C \parallel y \parallel, x, y \in B. \tag{2.5}$$

If F is any set of functionals q fulfilling (a), (b) and (c) with the same constant C, then (a), (b) et (c) are inherited by $g(x) = \sup\{q(x), q \in F\}$, which therfore satisfies (10).

Theorem 2 (Hamadouche, Taleb [4]) Let $(X_n)_n$ be a sequence of indepe-nent centred random variables non identically distributed. Suppose that there are $\gamma > 2$, $m > 0$ and $M > 0$ such that

$$m \leq \mathbb{E}|X_j|^2 \quad \text{and} \quad \mathbb{E}|X_j|^\gamma \leq M < +\infty, \forall j > 1.$$

Then the distributions of $s_n^{-1}\xi_n$ converge weakly to the Wiener measure P_W in H_α^0, for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$
The functionals $q_r(x) = \frac{\lambda_r(x)}{(r-r^-)^\alpha}$ satisfy the hypotheses of lemma 2 with the same constant $C = 1$. The same holds for $g_n = \max\limits_{1 \leq 2^j \leq n} \max\limits_{r \in D_j} q_r$ and $g(x) = \sup\limits_{j \geq 1} \max\limits_{r \in D_j} q_r(x)$.
(8), (9) and lemma 1 leads to

$$s_n^{-1} DI(n, \alpha) = g(s_n^{-1}\xi_n) + o_P(1),$$

the convergence of $s_n^{-1} DI(n, \alpha)$ to $DI(\alpha)$ is then given by theorem 2.

2.2 Consistency of $DI(n, \alpha)$

Let $(H'_A) : X_k = \begin{cases} m_c + X'_k & \text{si k} \in I_n = \{k^* + 1, ..., m^*\} \\ X'_k & \text{si k} \in I_n^c = \{1, ..., n\} \setminus I_n \end{cases}$

where $m_c \neq 0$ and X'_k satisfy (H'_0).

The consistency to reject (H'_0) against (H'_A) for larges values of $DI(n, \alpha)$ is given by the following result
Theorem 3 Write $l^* = m^* - k^*$ for the length of epidemics and assume that

$$\lim_{n \to +\infty} n s_n^{-1} h_n^{1-\alpha} |m_c| = +\infty, \tag{2.6}$$

where $h_n = min\{\frac{l^*}{n}, 1 - \frac{l^*}{n}\}$,
then under (H'_A)

$$s_n^{-1} DI(n,\alpha) \xrightarrow[n\to\infty]{P} +\infty.$$

Proof

First case : $\frac{l^*}{n} \leq \frac{1}{2}$ $(h_n = \frac{l^*}{n})$.

Let $S'_n = \sum_{i=1}^{n} X'_i$. Introducing the cardinals
$a_{n,r} = |I_n \cap]nr^-, nr]|$ and $b_{n,r} = |I_n \cap]nr, nr^+]|$.

we can write

$$\lambda_r(S_n) = \lambda_r(S'_n) + \frac{1}{2}(a_{n,r} - b_{n,r})m_c.$$

This implies

$$\begin{aligned} |\lambda_r(S_n)| &= |\frac{1}{2}(a_{n,r} - b_{n,r})m_c + \lambda_r(S'_n)| \\ &\geq \frac{1}{2}|a_{n,r} - b_{n,r}||m_c| - |\lambda_r(S'_n)|. \end{aligned}$$

Using Račkauskas and Suquet's result [9], we have

$$\max_{1\leq 2^j \leq n} \max_{r\in D_j} \frac{|a_{n,r} - b_{n,r}|}{2^{-j\alpha}} \geq \frac{n(\frac{l*}{n})}{2^{2\alpha+1}(\frac{l*}{n})^\alpha}.$$

which gives

$$DI(n,\alpha) \geq \frac{n(\frac{l*}{n})}{2^{2\alpha+2}(\frac{l*}{n})^\alpha}|m_c| - DI'(n,\alpha).$$

Hence

$$\begin{aligned} s_n^{-1}DI(n,\alpha) &\geq s_n^{-1}\frac{n(\frac{l^*}{n})}{2^{2\alpha+2}(\frac{l^*}{n})^\alpha}|m_c| - s_n^{-1}DI'(n,\alpha) \\ &= \frac{ns_n^{-1}h_n^{1-\alpha}}{2^{2\alpha+2}}|m_c| - s_n^{-1}DI'(n,\alpha). \end{aligned}$$

By theorem 1, $n^{-\frac{1}{2}}DI'(n,\alpha)$ is stochastically bounded. Using (16), the factor of $|m_c|$ goes to infinity, then

$$s_n^{-1} DI(n,\alpha) \xrightarrow[n\longrightarrow+\infty]{P} +\infty.$$

Second case : $\frac{l^*}{n} > \frac{1}{2}$ $(h_n = 1 - \frac{l^*}{n})$

a) If $t_{k^*} \geq 1 - t_{m^*}$ $\quad (t_{k^*} \geq (1 - \frac{l^*}{n})/2)$

There is a unique j such that $0 < 2^{-j-1} < t_{k^*} \leq 2^{-j} \leq 1/2 < t_{m^*}$. Let $r_0 = 2^{-j} \in D_j$, we obtain

$$
\begin{aligned}
2\lambda_{r_0}(S_n) &= \sum_{nr_0^- \leq k \leq nr_0} X_k - \sum_{nr \leq k \leq nr_0^+} X_k \\
&= \sum_{nr_0^- \leq k \leq nt_{k_*}} X_k^{'} + \sum_{nt_{k_*} \leq k \leq nr_0} (X_k^{'} + m_c) - \sum_{nr_0 \leq k \leq nr_0^+} (X_k^{'} + m_c) \\
&= [(nr_0 - nr_0^-) + (nr_0^- - nt_{k_*}) - (nr_0^+ - nr_0)]m_c + 2\lambda_{r_0}(S_n^{'}) \\
&= (nr_0^- - nt_{k_*})m_c + \lambda_{r_0}(S_n^{'}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
|\lambda_{r_0}(S_n)| &\geq \frac{1}{2}\left|nr_0^- - nt_{k_*}\right| \ |m_c| - \left|\lambda_{r_0}(S_n^{'})\right| \\
&\geq \frac{1}{4}n(1 - \frac{l^*}{n}) \, |m_c| - \left|\lambda_{r_0}(S_n^{'})\right|.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
DI(n,\alpha) &\geq \frac{1}{4}n(1 - \frac{l^*}{n})n^\alpha \, |m_c| - DI^{'}(n,\alpha). \\
&\geq \frac{1}{4}n \ (1 - \frac{l^*}{n})(1 - \frac{l^*}{n})^\alpha \, |m_c| - DI^{'}(n,\alpha).
\end{aligned}
$$

Finally

$$
s_n^{-1} DI(n,\alpha) \geq \frac{1}{4}n \ s_n^{-1} h_n^{1-\alpha} \, |m_c| - s_n^{-1} DI^{'}(n,\alpha).
$$

b) If $t_{k^*} < 1 - t_{m^*}$ $\quad (1 - t_{m^*} \geq (1 - \frac{l^*}{n})/2)$
We fix j by $1 - 2^{-j} \leq t_{m^*} < 1 - 2^{-j-1}$ and we choose $r_0 = 1 - 2^{-j} \in D_j$.
We have

$$
\begin{aligned}
2\lambda_{r_0}(S_n) &= \sum_{nr_0^- \leq k \leq nr_0} X_k - \sum_{nr \leq k \leq nr_0^+} X_k \\
&= \sum_{nr_0^- \leq k \leq nr_0} (X_k^{'} + m_c) - \sum_{nr_0 \leq k \leq nt_{m^*}} (X_k^{'} + m_c) - \sum_{nt_{m^*} \leq k \leq nr_0^+} X_k^{'} \\
&= (nr_0 - nr_0^- - nt_{m_*} + nr_0) \ m_c + 2\lambda_{r_0}(S_n^{'}) \\
&= n(1 - t_{m^*})m_c + 2\lambda_{r_0}(S_n^{'}).
\end{aligned}
$$

Then

$$\begin{aligned}
|\lambda_{r_0}(S_n)| &\geq \frac{1}{2}n(1-t_{m^*})\ |m_c| - \left|\lambda_{r_0}(S_n^{'})\right| \\
&\geq \frac{n}{4}(1-\frac{l^*}{n})\,|m_c| - \left|\lambda_{r_0}(S_n^{'})\right|.
\end{aligned}$$

This implies that

$$DI(n,\alpha) \geq \frac{1}{4}n\ (1-\frac{l^*}{n})^{1-\alpha}\ \ |m_c| - DI^{'}(n,\alpha).$$

Hence

$$s_n^{-1}DI(n,\alpha) \geq \frac{1}{4}n\ s_n^{-1}h_n^{1-\alpha}\,|m_c| - s_n^{-1}DI^{'}(n,\alpha).$$

By theorem 1, $n^{-\frac{1}{2}}DI'(n,\alpha)$ is stochastically bounded. Using (16), the factor of $|m_c|$ goes to infinity, then

$$s_n^{-1}DI(n,\alpha) \underset{n\longrightarrow+\infty}{\overset{P}{\longrightarrow}} +\infty.$$

2.3 Application

As an example of applications of theorem 1 in the case of a sequence of i.i.d. random variables, Račkauskas and Suquet [9] considered change point problems for distribution function, characteristic function as well as changes in covariance matrices.

Our objectif here is to present an application of theorem 1 to detect epidemic change in the variance for a sequence of independent not identically distributed random variables.

Testing change of the variance : Let $(X_1, X_2, ..., X_n)$ be a sample of random variables with variances $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$ respectivly.

We want to test the null hypothesis: $(H_0) : \sigma_1^2 = \sigma_2^2 = ... = \sigma_n^2 = \widetilde{\sigma}^2$, against the epidemic aternative

$(H_A) : \exists 1 < k^* < m^* < n$ such that

$\sigma_1^2 = ... = \sigma_{k^*}^2 = \sigma_{m^*+1}^2 = ... = \sigma_n^2, \sigma_{k^*+1}^2 = ... = \sigma_{m^*}^2$ and $\sigma_{k^*}^2 \neq \sigma_{k^*+1}^2$.

Define

$$V_n(s) = \sum_{1\leq k\leq ns} (X_k^2 - \widetilde{\sigma}^2), s \in [0,1].$$

We consider the test statistic

$$\nu(n,\alpha) = \max_{1\leq 2^j\leq n} \frac{1}{2^{-j\alpha}} \max_{r\in D_j} |\lambda_r(V_n)|.$$

We note $\Psi(u) = P(\lambda_r(W) \leq u)$ and $\overline{s}_n^2 = \sum_{i=1}^{n}(EX_i^4 - \widetilde{\sigma}^4)$. Under the hypothesis (H_0'), we have

Theorem 4 Suppose there are $\gamma > 4$, $m > 0$ and $M > 0$ such that

$$\tilde{\sigma}^4 < m \leq E|X_j|^4 \text{ and } E|X_j|^\gamma \leq M < +\infty, \forall j > 1.$$

Assume that

$$\sum_{i=1}^{\infty} \mathbb{E}(X_i^4) = 0. \tag{2.7}$$

Then for all $\alpha < \frac{1}{2} - \frac{2}{\gamma}$, we have

$$\lim_{n \longrightarrow +\infty} P\{\overline{s}_n^{-1}\nu(n,\alpha) \leq u\} = \Psi_\alpha(u), \forall u > 0,$$

where $\Psi_\alpha(u) = \prod_{j=1}^{\infty}[\Psi(2^{j\alpha}\sqrt{u})]^{2^{j-1}}$.

Proof. We use theorem 1 with the random variables $Y_1, Y_2, ..., Y_n$ defined by

$Y_k = X_k^2 - \widetilde{\sigma}^2$, $k = 1, ..., n$.

We have

$$\mathbb{E}|Y_i|^2 = \mathbb{E}|X_i^2 - \widetilde{\sigma}^2|^2 = \mathbb{E}|X_i|^4 - 2\widetilde{\sigma}^2\mathbb{E}|X_i|^2 + \widetilde{\sigma}^4.$$

Under (H_0'), X_i are centred, then $E|X_i|^2 = \widetilde{\sigma}^2$ under (H_0).

this imply

$$\mathbb{E}|Y_i|^2 = \mathbb{E}|X_i|^4 - \widetilde{\sigma}^4 \geq m - \widetilde{\sigma}^4,$$

with $m - \widetilde{\sigma}^4 > 0$.

We have also

$$\mathbb{E}|Y_i|^{\frac{\gamma}{2}} = \mathbb{E}|X_i^2 - \widetilde{\sigma}^2|^{\frac{\gamma}{2}} \leq \mathbb{E}(|X_i|^2 + \widetilde{\sigma}^2)^{\frac{\gamma}{2}}$$

Using Jensen's inequality, we have

$$\mathbb{E}(|X_i|^2 + \widetilde{\sigma}^2)^{\frac{\gamma}{2}} \leq 2^{\frac{\gamma}{2}-1}[\mathbb{E}|X_i|^\gamma + \widetilde{\sigma}^\gamma] \leq 2^{\frac{\gamma}{2}-1}[M + \widetilde{\sigma}^\gamma].$$

For $\frac{\gamma}{2} > 2$ and $2^{\frac{\gamma}{2}-1}[M + \widetilde{\sigma}^\gamma] > 0$,

$$\mathbb{E}|Y_i|^{\frac{\gamma}{2}} \leq 2^{\frac{\gamma}{2}-1}[M + \widetilde{\sigma}^\gamma] < +\infty, \forall i \geq 1.$$

We have

$$\forall A > 0,\ P(|Y_i| > A\frac{\overline{s}_n}{n^{\frac{1}{2}}}) = P(|X_i^2 - \widetilde{\sigma}^2| > A\frac{\overline{s}_n}{n^{\frac{1}{2}}}) \leq \frac{\mathbb{E}|X_i^2 - \widetilde{\sigma}^2|^2}{(A\frac{\overline{s}_n}{n^{\frac{1}{2}}})^2}.$$

Then

$$\forall A>0,\ \sum_{i=1}^{n} P(|Y_i|>A\frac{\overline{s}_n}{n^{\frac{1}{2}}}) \leq \frac{\sum_{i=1}^{n}\mathbb{E}|X_i^2-\widetilde{\sigma}^2|^2}{(A\frac{\overline{s}_n}{n^{\frac{1}{2}}})^2} \leq \frac{\sum_{i=1}^{n}\mathbb{E}(X_i^4)}{(A\frac{\overline{s}_n}{n^{\frac{1}{2}}})^2}.$$

In the other hand

$$A\frac{\overline{s}_n}{n^{\frac{1}{2}}} = A\frac{\sqrt{\sum_{i=1}^{n}(\mathbb{E}X_i^4-\widetilde{\sigma}^4)}}{n^{\frac{1}{2}}} \geq A\frac{\sqrt{n(m-\widetilde{\sigma}^4)}}{n^{\frac{1}{2}}} = A\sqrt{m-\widetilde{\sigma}^4}>0.$$

Consequently

$$\forall A>0,\ \sum_{i=1}^{n} P(|Y_i|>A\frac{\overline{s}_n}{n^{\frac{1}{2}}}) \leq \frac{\sum_{i=1}^{n}\mathbb{E}(X_i^4)}{A^2(m-\widetilde{\sigma}^4)^2}.$$

Using (12), we get

$$\forall A>0, \lim_{n\longrightarrow+\infty}\sum_{i=1}^{n} P(|Y_i|>A\frac{\overline{s}_n}{n^{\frac{1}{2}}}) = 0.$$

Conditions of theorem 1 are satisfied for $(Y_n)_{n\geq 1}$ with $\frac{\gamma}{2}>2$, $m-\widetilde{\sigma}^4>0$ and $2^{\frac{\gamma}{2}-1}[M+|\widetilde{\sigma}|^{\gamma}]>0$.
Then, $\forall\alpha<\frac{1}{2}-\frac{1}{\frac{\gamma}{2}}=\frac{1}{2}-\frac{2}{\gamma}$,

$$\overline{s}_n^{-1}\nu(n,\alpha) \underset{n\longrightarrow\infty}{\overset{\mathcal{D}}{\longrightarrow}} DI(\alpha),$$

Therfore

$$\lim_{n\longrightarrow+\infty} P\{\overline{s}_n^{-1}\nu(n,\alpha)\leq u\} = P\{DI(\alpha)\leq u\}, \forall u>0$$

and

$$P\{DI(\alpha)\leq u\} = P\{\sup_{j\geq 1}\frac{1}{2^{-j\alpha}}\max_{r\in D_j}|\lambda_r(W)|\leq u\}.$$

Since the random variables $\lambda_r(W)$ are i.i.d.,

$$\begin{aligned}
P\{\sup_{j\geq 1}\frac{1}{2^{-j\alpha}}\max_{r\in D_j}|\lambda_r(W)|\leq u\} &= \prod_{j=1}^{\infty} P(\frac{1}{2^{-j\alpha}}\max_{r\in D_j}|\lambda_r(W)|\leq u) \\
&= \prod_{j=1}^{\infty} P(\max_{r\in D_j}|\lambda_r(W)|\leq 2^{j\alpha}u) \\
&= \prod_{j=1}^{\infty} [P(|\lambda_r(W)|\leq 2^{j\alpha}u)]^{2^j-1} \\
&= \prod_{j=1}^{\infty} [\Psi(2^{j\alpha}u)]^{2^j-1}.
\end{aligned}$$

Let $\Psi_\alpha(u) = \prod_{j=1}^{\infty}[\Psi(2^{j\alpha}u)]^{2^{j-1}}$, then

$$\lim_{n \longrightarrow +\infty} P\{\overline{s}_n^{-1}\nu(n,\alpha) \leq u\} = \Psi_\alpha(u), \forall u > 0.$$

Let us denote under (H_A) by $\overline{\sigma}^2$ the variance during the epidemics, we have the following consistency result

Theorem 5

Suppose that (12) is satisfied. Then under (H_A), we have

$$\overline{s}_n^{-1}\nu(n,\alpha) \underset{n \longrightarrow \infty}{\overset{P}{\longrightarrow}} +\infty.$$

Proof

We shall verify conditions of theorem 3 for $Y_i = X_i^2 - \widetilde{\sigma}^2$ and $m_c = \overline{\sigma}^2 - \widetilde{\sigma}^2$.

$$n\overline{s}_n^{-1}h_n^{1-\alpha} = \frac{nh_n^{1-\alpha}}{\sqrt{\sum_{i=1}^{n}\mathbb{E}(X_i^4 - \widetilde{\sigma}^4)}} \geq \frac{nh_n^{1-\alpha}}{\sqrt{\sum_{i=1}^{n}\mathbb{E}X_i^4}}.$$

By (12), the minorant of $n\overline{s}_n^{-1}h_n^{1-\alpha}$ goes to $+\infty$, condition (11) is satisfied for Y_i.

We have also

$$Y_i = X_i^2 - \widetilde{\sigma}^2 = \begin{cases} (\overline{\sigma}^2 - \widetilde{\sigma}^2) + X_i^2 - \overline{\sigma}^2 & \text{if } i \in I_n = \{k^* + 1, ..., m^*\} \\ X_i^2 - \widetilde{\sigma}^2 & \text{if } i \in I_n^c = \{1, ..., n\} \setminus I_n \end{cases}$$

Let

$$Y_i' = \begin{cases} X_i^2 - \overline{\sigma}^2 & \text{if } i \in I_n \\ X_i^2 - \widetilde{\sigma}^2 & \text{if } i \in I_n^c \end{cases}$$

Then

$$Y_i = \begin{cases} m_c + Y_i' & \text{if } i \in I_n \\ Y_i' & \text{if } i \in I_n^c \end{cases}$$

Under (H_A), $m_c = \overline{\sigma}^2 - \widetilde{\sigma}^2 \neq 0$ and the random variables Y_i' are independent and centred. Then the Y_i are under (H_A').
By theorem 3, we have

$$\overline{s}_n^{-1}\nu(n,\alpha) \underset{n \longrightarrow \infty}{\overset{P}{\longrightarrow}} +\infty.$$

3 Stationary dependent case (α -mixing)

3.1 Convergence of $DI(n,\alpha)$

Let $(X_n)_{n\geq 1}$ a sequence of random variables.
Under the hypothesis

(H_0') : $(X_n)_n$ is strictly stationary mixing sequence with mean 0, we have the result

Theorem 6 Under (H_0), suppose that there exist $\gamma > 2$ and $\epsilon > 0$ such that

$$E\,|X_1|^{\gamma+\epsilon} < +\infty \text{ and } \sum_{n=1}^{\infty}(n+1)^{\frac{\gamma}{2}-1}(\alpha_n)^{\frac{\epsilon}{\gamma+\epsilon}} < +\infty.$$

Then for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$,

$$\sigma^{-1}n^{-\frac{1}{2}}DI(n,\alpha) \xrightarrow[n\longrightarrow\infty]{D} DI(\alpha),$$

where $\sigma^2 = \mathbb{E}(X_1^2) + 2\sum_{j=2}^{\infty} cov(X_1, X_j) < +\infty$.

Proof

First we observe that

$$DI(n,\alpha) = \max_{1\le 2^{j\alpha}\le n} \frac{1}{2^{-j\alpha}} \max_{r\in D_j} |\lambda_r(S_{n.})|\,, \tag{3.1}$$

where $(S_{n.})$ is the discontinuous process $(S(nt), 0 \le t \le 1)$ defined by

$$S(nt) = \sum_{k=1}^{[nt]} X_k.$$

This process may be written as

$$S(nt) = \xi_n(t) - (nt - [nt])X_{[nt]+1},$$

from which we see that

$$|\lambda_r(S_{n.}) - \lambda_r(\xi_n)| \le 2\max_{1\le i\le n} |X_i|\,.$$

Using this estimation in (4), we obtain

$$\sigma^{-1}n^{-\frac{1}{2}}DI(n,\alpha) = g_n(\sigma^{-1}n^{-\frac{1}{2}}\xi_n) + Z_n, \tag{3.2}$$

where $g_n(x) = \max\limits_{1\le 2^{j\alpha}\le n} \max\limits_{r\in D_j} \frac{|\lambda_r(x)|}{2^{-j\alpha}}$, $x \in H_\alpha^0$, and the random variable Z_n satisfies

$$|Z_n| \le \frac{2}{\sigma n^{\frac{1}{2}}(\frac{1}{n})^{\alpha}} \max_{1\le i\le n} |X_i| = \frac{2}{\sigma n^{\frac{1}{2}-\alpha}} \max_{1\le i\le n} |X_i|\,. \tag{3.3}$$

We have

$$P(\frac{1}{\sigma n^{\frac{1}{2}-\alpha}} \max_{1\le i\le n} |X_i| > \varepsilon) = P(\max_{1\le i\le n} |X_i| > \varepsilon\sigma n^{\frac{1}{2}-\alpha}) \le \sum_{i=1}^{n} P(|X_i| > \varepsilon\sigma n^{\frac{1}{2}-\alpha}).$$

Then

$$P(\frac{1}{\sigma n^{\frac{1}{2}-\alpha}} \max_{1\leq i\leq n} |X_i| > \varepsilon) \leq nP(|X_1| > \varepsilon\sigma n^{\frac{1}{2}-\alpha}).$$

In the other hand

$$\forall p,\ t^p P(|X_1| > t) \leq t^p \frac{\mathbb{E}\,|X_1|^{\gamma+\epsilon}}{t^{\gamma+\varepsilon}} = t^{p-\gamma-\varepsilon}\ \mathbb{E}\,|X_1|^{\gamma+\epsilon}.$$

Since $\mathbb{E}\,|X_1|^{\gamma+\epsilon} < \infty$, then for $p = (\frac{1}{2} - \alpha)^{-1}$ $t^p P(|X_1| > t)$ goes to 0 for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$. This implies $P(|X_1| > t) = o(t^{-p})$. Hence, we have the convergence in probability to 0 of $\max\limits_{1\leq i\leq n} \frac{|X_i|}{n^{\frac{1}{2}-\alpha}}$.

The lemmas 1, 2 and the following result are useful to simpify the proof of our theorem

Theorem 7 (Hamadouche [3]) Let $(X_n)_n$ be a strictly stationary $\alpha-$mixing sequence of centred random variables. Suppose that there are $\gamma > 2$ and $\epsilon > 0$ such that

$$E\,|X_1|^{\gamma+\epsilon} < +\infty \quad \text{and} \sum_{n=1}^{\infty}(n+1)^{\frac{\gamma}{2}-1}(\alpha_n)^{\frac{\epsilon}{\gamma+\epsilon}} < +\infty.$$

Then the distributions of $\sigma^{-1}n^{-\frac{1}{2}}\xi_n$, defined in (1) converge weakly to the Wiener measure P_W in H^0_α, for all $\alpha < \frac{1}{2} - \frac{1}{\gamma}$.

The functionals $q_r(x) = \frac{\lambda_r(x)}{(r-r^-)^\alpha}$ satisfy the hypotheses of lemma 2 with the same constant $C = 1$. The same holds for $g_n = \max\limits_{1\leq 2^{j\alpha}\leq n} \max\limits_{r\in D_j} q_r$ and $g(x) = \sup\limits_{j\geq 1}\max\limits_{r\in D_j} q_r(x)$.

With (14) and (15), lemma 1 leads to

$$\sigma^{-1}n^{-\frac{1}{2}}DI(n,\alpha) = g(\sigma^{-1}n^{-\frac{1}{2}}\xi_n) + o_P(1)$$

and the convergence of $\sigma^{-1}n^{-\frac{1}{2}}DI(n,\alpha)$ to $DI(\alpha)$ is then given by theorem 7.

3.2 Consistency of $DI(n,\alpha)$

Let $(H'_A) : X_k = \begin{cases} m_c + X'_k & \text{si k} \in I_n = \{k^* + 1, ..., m^*\} \\ X'_k & \text{si k} \in I_n^c = \{1, ..., n\} \setminus I_n \end{cases}$

where $m_c \neq 0$ and X'_k satisfy (H'_0).

The consistency to reject (H'_0) against (H'_A) for larges values of $DI(n,\alpha)$ is given by the following result

Theorem 8 Write $l^* = m^* - k^*$ for the length of epidemics and assume that

$$\lim_n n^{\frac{1}{2}} |m_c| h_n^{1-\alpha} = \infty,$$

where $h_n = min\{\frac{l^*}{n}, 1 - \frac{l^*}{n}\}$,
then under (H'_A),

$$n^{-\frac{1}{2}} DI(n,\alpha) \xrightarrow[n\to\infty]{P} \infty.$$

Proof. Using the proof of theorem 3, We have
First case : $\frac{l^*}{n} \leq \frac{1}{2}$ $(h_n = \frac{l^*}{n})$.

$$DI(n,\alpha) \geq \frac{n(\frac{l*}{n})}{2^{2\alpha+2}(\frac{l*}{n})^{\alpha}} |m_c| - DI'(n,\alpha).$$

Consequently

$$n^{-\frac{1}{2}} DI(n,\alpha) \geq \frac{n^{\frac{1}{2}} h_n^{1-\alpha}}{2^{2\alpha+2}} |m_c| - n^{-\frac{1}{2}} DI'(n,\alpha).$$

By theorem 6, $n^{-\frac{1}{2}} DI'(n,\alpha)$ is stochastically bounded. Using (16), the factor of $|m_c|$ goes to infinity, then

$$n^{-\frac{1}{2}} DI(n,\alpha) \xrightarrow[n\longrightarrow+\infty]{P} +\infty.$$

Second case : $\frac{l^*}{n} > \frac{1}{2}$ $(h_n = 1 - \frac{l^*}{n})$
a) If $t_{k^*} \geq 1 - t_{m^*}$ $\quad (t_{k^*} \geq (1 - \frac{l^*}{n})/2)$,

$$n^{-\frac{1}{2}} DI(n,\alpha) \geq \frac{1}{4} n^{\frac{1}{2}} h_n^{1-\alpha} |m_c| - n^{-\frac{1}{2}} DI^{'}(n,\alpha).$$

b) If $t_{k^*} < 1 - t_{m^*}$ $\quad (1 - t_{m^*} \geq (1 - \frac{l^*}{n})/2)$,

$$n^{-\frac{1}{2}} DI(n,\alpha) \geq \frac{1}{4} n^{\frac{1}{2}} h_n^{1-\alpha} |m_c| - n^{-\frac{1}{2}} DI^{'}(n,\alpha).$$

By theorem 2, $n^{-\frac{1}{2}} DI'(n,\alpha)$ is stochastically bounded. Using (16), the factor of $|m_c|$ goes to infinity, then

$$n^{-\frac{1}{2}} DI(n,\alpha) \xrightarrow[n\longrightarrow+\infty]{P} +\infty.$$

3.3 Application

Testing change of the variance : Let $(X_1, X_2, ..., X_n)$ be a sample of random variables with variances $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$ respectivly.
We want to test the null hypothesis: $(H_0) : \sigma_1^2 = \sigma_2^2 = ... = \sigma_n^2 = \widetilde{\sigma}^2$,
against the epidemic aternative
$(H_A) : \exists 1 < k^* < m^* < n$ such that
$\sigma_1^2 = ... = \sigma_{k^*}^2 = \sigma_{m^*+1}^2 = ... = \sigma_n^2, \sigma_{k^*+1}^2 = ... = \sigma_{m^*}^2$ and $\sigma_{k^*}^2 \neq \sigma_{k^*+1}^2$.
Define

$$V_n(s) = \sum_{1 \leq k \leq ns} (X_k^2 - \widetilde{\sigma}^2), s \in [0, 1].$$

We consider the test statistic

$$\nu(n, \alpha) = \max_{1 \leq 2^j \leq n} \frac{1}{2^{-j\alpha}} \max_{r \in D_j} |\lambda_r(V_n)| .$$

We note

$$\Psi(u) = P(|\lambda_r(W)| \leq u).$$

Under the hypothesis (H_0') we have

Theorem 9 Suppose there are $\gamma > 4$ and $\epsilon > 0$ such that $E|X_1|^{\gamma+\epsilon} < +\infty$ and

$$\sum_{n=1}^{\infty} (n+1)^{\frac{\gamma}{4}-1} (\alpha_n)^{\frac{\epsilon}{\gamma+\epsilon}} < +\infty.$$

Then for all $\alpha < \frac{1}{2} - \frac{2}{\gamma}$, we have

$$\lim_{n \longrightarrow \infty} P\{\sigma^{-2} n^{-1} \nu^2(n, \alpha) \leq u\} = \Psi_\alpha(u), \forall u > 0,$$

where $\Psi_\alpha(u) = \prod_{j=1}^{\infty} [\Psi(2^{j\alpha}\sqrt{u})]^{2^{j-1}}$.

Proof. We use theorem 6 with the random variables $Y_1, Y_2, ..., Y_n$ defined by

$Y_k = X_k^2 - \widetilde{\sigma}^2, k = 1, ..., n.$
Under (H_0'), $(Y_n)_n$ is a strictly stationary $\alpha-$mixing sequence with mixing coeﬃcient $\widehat{\alpha}_n \leq \alpha_n$.
For $\gamma > 4$ and $\epsilon > 0$

$$|Y_1|^{\frac{\gamma}{2}+\frac{\epsilon}{2}} = \left|X_1^2 - \widetilde{\sigma}^2\right|^{\frac{\gamma}{2}+\frac{\epsilon}{2}} \leq (|X_1^2| + \widetilde{\sigma}^2)^{\frac{\gamma}{2}+\frac{\epsilon}{2}}.$$

Using Jensen's inequality, we have

$$(|X_1^2| + \widetilde{\sigma}^2)^{\frac{\gamma}{2}+\frac{\epsilon}{2}} \leq 2^{\frac{\gamma}{2}+\frac{\epsilon}{2}-1}[|X_1|^{\gamma+\epsilon} + \widetilde{\sigma}^{\gamma+\epsilon}]$$

then

$$\mathbb{E}|Y_1|^{\frac{\gamma}{2}+\frac{\epsilon}{2}} \leq 2^{\frac{\gamma}{2}+\frac{\epsilon}{2}-1}[\mathbb{E}|X_1|^{\gamma+\epsilon} + \widetilde{\sigma}^{\gamma+\epsilon}]$$

Since $\mathbb{E}\,|X_1|^{\gamma+\epsilon} < +\infty$, then $\mathbb{E}\,|Y_1|^{\frac{\gamma}{2}+\frac{\epsilon}{2}} < +\infty$.
We have also

$$\sum_{n=1}^{\infty}(n+1)^{\frac{\frac{\gamma}{2}}{2}-1}(\widehat{\alpha}_n)^{\frac{\frac{\epsilon}{2}}{\frac{\gamma}{2}+\frac{\epsilon}{2}}} \leq \sum_{n=1}^{\infty}(n+1)^{\frac{\gamma}{4}-1}(\alpha_n)^{\frac{\epsilon}{\gamma+\epsilon}} < +\infty.$$

Conditions of theorem 9 are satisfied for $(Y_n)_{n\geq 1}$ with $\gamma > 4$ and $\epsilon > 0$.
Then for all $\alpha < \frac{1}{2} - \frac{1}{\frac{\gamma}{2}} = \frac{1}{2} - \frac{2}{\gamma}$ and $\alpha < \frac{1}{2} - \frac{2}{\gamma+\epsilon}$ i.e for all $\alpha < \min(\frac{1}{2} - \frac{2}{\gamma}, \frac{1}{2} - \frac{2}{\gamma+\epsilon}) = \frac{1}{2} - \frac{2}{\gamma}$, we have

$$\sigma^{-1}n^{-\frac{1}{2}}\nu(n,\alpha) \xrightarrow[n\longrightarrow\infty]{\mathcal{L}} DI(\alpha),$$

therfore

$$\lim_{n\longrightarrow+\infty} P\{\sigma^{-1}n^{-\frac{1}{2}}\nu(n,\alpha) \leq u\} = P\{DI(\alpha) \leq u\}, \forall u > 0,$$

and

$$P\{DI(\alpha) \leq u\} = P\{\sup_{j\geq 1}\frac{1}{2^{-j\alpha}}\max_{r\in D_j}|\lambda_r(W)| \leq u\}.$$

Since random variables $\lambda_r(W)$ are i.i.d., we have

$$\begin{aligned}
P\{\sup_{j\geq 1}\frac{1}{2^{-j\alpha}}\max_{r\in D_j}|\lambda_r(W)| \leq u\} &= \prod_{j=1}^{\infty} P(\frac{1}{2^{-j\alpha}}\max_{r\in D_j}|\lambda_r(W)| \leq u) \\
&= \prod_{j=1}^{\infty} P(\max_{r\in D_j}|\lambda_r(W)| \leq 2^{j\alpha}u) \\
&= \prod_{j=1}^{\infty}[P(|\lambda_r(W)| \leq 2^{j\alpha}u)]^{2^j-1} \\
&= \prod_{j=1}^{\infty}[\Psi(2^{j\alpha}u)]^{2^j-1}.
\end{aligned}$$

Let $\Psi_\alpha(u) = \prod_{j=1}^{\infty}[\Psi(2^{j\alpha}u)]^{2^{j-1}}$, then

$$\lim_{n\longrightarrow+\infty} P\{\sigma^{-1}n^{-\frac{1}{2}}\nu(n,\alpha) \leq u\} = \Psi_\alpha(u), \forall u > 0.$$

Let us denote under (H_A) by $\overline{\sigma}^2$ the variance during the epidemics, we have the following consistency result

Theorem 10 Let $u_n = n^{\frac{1}{2}}h_n^{1-\alpha}$. Under (H_A), we suppose that

$$\lim_{n\longrightarrow\infty} u_n^2(\overline{\sigma}^2 - \widetilde{\sigma}^2)^2 = \infty. \tag{3.4}$$

Then

$$n^{-1}\nu^2(n,\alpha) \xrightarrow[n\longrightarrow\infty]{P} \infty.$$

Proof.

We verify conditions of theorem 8 for the random variables $Y_i = X_i^2 - \widetilde{\sigma}^2$. Condition (16) is fulfiled with $m_c = \overline{\sigma}^2 - \widetilde{\sigma}^2$.
We have

$$Y_i = X_i^2 - \widetilde{\sigma}^2 = \begin{cases} (\overline{\sigma}^2 - \widetilde{\sigma}^2) + X_i^2 - \overline{\sigma}^2 & \text{si } i \in I_n = \{k^* + 1, ..., m^*\} \\ X_i^2 - \widetilde{\sigma}^2 & \text{si } i \in I_n^c = \{1, ..., n\} \setminus I_n \end{cases}$$

Let $Y_i' = \begin{cases} X_i^2 - \overline{\sigma}^2 & \text{si } i \in I_n \\ X_i^2 - \widetilde{\sigma}^2 & \text{si } i \in I_n^c \end{cases}$

Then

$$Y_i = \begin{cases} m_c + Y_i' & \text{si } i \in I_n \\ Y_i' & \text{si } i \in I_n^c \end{cases}$$

Under (H_A), $m_c = \overline{\sigma}^2 - \widetilde{\sigma}^2 \neq 0$ and Y_i' are strictly stationary mixing variables with mean 0 . By theorem 8, we have

$$n^{-\frac{1}{2}}\nu(n,\alpha) \xrightarrow[n\longrightarrow\infty]{P} +\infty.$$

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