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A remark on the $\mathcal{L}_1$-norm of Brownian motion*

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Abstract

The Laplace transform of the integral of the absolute value of a real Brownian motion has been computed in 1945 by M. Kac [5], with the help of a long and subtle asymptotic analysis on Bessel functions. Up to now, there does not seem to exist a shorter proof of this well-known computation. In this semi-historical note we observe that Kac's argument could have been greatly simplified back then, had he used the eigenfunction expansion associated to a real Schrödinger operator with linear potential, evaluated in 1944 (and in the same linguistic part of the world) by R.-P. Bell [2].

Résumé

La transformée de Laplace de l'intégrale de la valeur absolue d'un mouvement brownien a été calculée en 1945 par M. Kac [5], à l'aide d'une longue et subtile analyse asymptotique sur les fonctions de Bessel. À ce jour, il ne semble pas exister une preuve plus courte pour ce calcul bien connu. Dans cette note semi-historique nous remarquons que l'argument de Kac aurait pu être grandement simplifié s'il avait utilisé le développement en série de fonctions propres associé à un opérateur de Schrödinger réel avec potentiel linéaire, calculé en 1944 (et dans la même partie linguistique du monde) par R.-P. Bell [2].

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1 Recalls on the Laplace transforms of $\mathcal{L}_p$-Brownian functionals

Let $\{W_t,\ t \geq 0\}$ be a standard linear Brownian motion and denote by $\mathbb{P}^x$ its law starting from $x \in \mathbb{R}$, setting $\mathbb{P}^0 = \mathbb{P}$ for simplicity. Consider for every $p > 0$ the $\mathcal{L}_p$-functionals

$$||W||_p \ = \ \left(\int_0^1 |W_t|^p dt \right)^{\frac{1}{p}}.$$

Introducing the function

$$u_p(t,x) \ = \ \mathbb{E}^x \left[\exp - \int_0^t |W_s|^p ds \right] \ = \ \mathbb{E}^x \left[\mathrm{e}^{-t^{p/2+1}||W||_p^p} \right] \qquad t \geq 0,\ x \in \mathbb{R},$$

a version of the Feynman-Kac formula which can be found e.g. in [6] Proposition 5.8, entails that it solves the PDE

$$\frac{\partial u_p}{\partial t}(t,x) \ = \ \frac{1}{2} \frac{\partial^2 u_p}{\partial x^2}(t,x) \ - \ |x|^p u_p(t,x)$$

with initial condition $u_p(0,x) = 1$. On the other hand, it is known - see Chapter V in [9] - that the self-adjoint negative operator

$$\mathcal{A}_p : f \ \mapsto \ \frac{1}{2} f'' \ - \ |x|^p f$$

acting on real $\mathcal{C}^2$ functions has a discrete and simple spectrum, and that there exists an orthonormal basis of $\mathcal{L}_2(\mathbb{R})$ made out of the corresponding eigenfunctions. In other words, there exists a sequence $0 < \lambda_1^p \leq \lambda_2^p \leq \ldots \leq \lambda_n^p \to +\infty$ and a sequence of fonctions $\{\psi_n^p,\ n \geq 1\}$ such that for any $m, n \geq 1$ one has $< \psi_n^p, \psi_m^p > = \delta_{mn}$ (with Kronecker's notation) and $\mathcal{A}_p \psi_n^p \ = \ -\lambda_n^p \psi_n^p$, and such that any $f \in \mathcal{L}_2(\mathbb{R})$ has the decomposition

$$f \ = \ \sum_{n \geq 1} < f, \psi_n^p > \psi_n^p \tag{1}$$

in $\mathcal{L}_2$, where $<,>$ stands for the usual scalar product. It is easy to see from basic properties of Brownian Motion that the function $x \mapsto u_p(t,x)$ is $\mathcal{C}^\infty$ with all its derivatives in $\mathcal{L}_2$. By Theorem 2.7 (i) in [9], this entails that (1) applied to $u_p(t,.)$ is also true in $\mathcal{L}_\infty$: one has

$$u_p(t,x) \ = \ \sum_{n \geq 1} \varphi_n^p(t) \psi_n^p(x)$$

for every $t \geq 0$ and $x \in \mathbb{R}$, where the φ_n^p are defined by the formula $\varphi_n^p(t) = < u_p(t,.), \psi_n^p >$. Differentiating under the integral yields

$$\begin{aligned}
\frac{d\varphi_n^p}{dt}(t) &= \int_{\mathbb{R}} \frac{\partial u_p}{\partial t}(t,x)\psi_n^p(x)dx &&= \int_{\mathbb{R}} \mathcal{A}_p u_p(t,x)\psi_n^p(x)dx \\
& &&= \int_{\mathbb{R}} \left(\sum_{n\geq 1} \varphi_n^p(t)\mathcal{A}_p\psi_n^p(x) \right) \psi_n^p(x)dx \\
& &&= -\int_{\mathbb{R}} \left(\sum_{n\geq 1} \lambda_n^p \varphi_n^p(t)\psi_n^p(x) \right) \psi_n^p(x)dx \\
& &&= -\lambda_n^p \varphi_n^p(t),
\end{aligned}$$

whence $\varphi_n^p(t) = c_n^p \mathrm{e}^{-\lambda_n^p t}$ for every $t \geq 0$. Recalling that $u_p(0,x) = 1$ for every $x \in \mathbb{R}$, we finally compute

$$c_n^p = \int_{\mathbb{R}} \psi_n^p(y)dy$$

which yields

$$u_p(t,x) = \sum_{n\geq 1} \mathrm{e}^{-\lambda_n^p t} \left(\int_{\mathbb{R}} \psi_n^p(y)dy \right) \psi_n^p(x).$$

Taking $x = 0$ and changing the variable, we can now write down the formula for the Laplace transforms of the $\mathcal{L}_p$-Brownian functionals:

$$\mathbb{E}\left[\mathrm{e}^{-t||W||_p^p} \right] = \sum_{n\geq 1} \mathrm{e}^{-\lambda_n^p t^{\frac{2}{p+2}}} \left(\int_{\mathbb{R}} \psi_n^p(y)dy \right) \psi_n^p(0), \quad t \geq 0. \tag{2}$$

This formula is very well-known, see e.g. the introduction of [3].

2 Specifying to $p = 1$

The eigenelements $\{\psi_n, \lambda_n\}$ of the operator $f \mapsto f'' - |x|f$ have been evaluated in [2], a paper seemingly unnoticed by Kac at the time. For the probabilistic operator $\mathcal{A}_1$, a change of variable and the normalization $< \psi_n^1, \psi_n^1 > = 1$ yield

$$\lambda_n^1 = 2^{-1/3}\lambda_n \qquad \text{and} \qquad \psi_n^1(x) = 2^{1/6}\psi_n(2^{1/3}x).$$

In view of (2), it is enough to consider the even eigenfunctions and the corresponding eigenvalues, and we will relabel them by $\{\psi_n, \lambda_n\}$ for simplicity. By (5) in [2], the λ_n are the successive zeroes of the function

$$x \mapsto J_{2/3}(\zeta) - J_{-2/3}(\zeta),$$

where J_α stands for the Bessel function of the first kind with index α and $\zeta = 2x^{3/2}/3$. By e.g. (10.4.17) in [1], we see that the λ_n are the successive zeroes of $x \mapsto \mathrm{Ai}'(-x)$, where

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(tx + t^3/3)dt$$

is the Airy function. For the corresponding eigenfunction ψ_n, equations (3) and (10) in [2] give

$$\psi_n^1(0) \;=\; 2^{1/6}\psi_n(0) \;=\; \frac{2^{-1/3}}{\sqrt{\lambda_n}}\cdot$$

We last compute

$$\begin{aligned}\int_{\mathbb{R}} \psi_n^1(y)dy \;=\; 2^{-1/6}\int_{\mathbb{R}} \psi_n(y)dy \;&=\; 2^{5/6}\int_0^\infty \psi_n(y)dy\\ &=\; 2^{5/6}\left(\int_0^{\lambda_n} \psi_n(y)dy \;+\; \int_{\lambda_n}^\infty \psi_n(y)dy\right).\end{aligned}$$

Again, by (3) and (10) in [2] and using the notation $\zeta_n = 2\lambda_n^{3/2}/3$, the first integral is given by

$$\begin{aligned}\int_0^{\lambda_n} \psi_n(y)dy &= \int_0^{\lambda_n}\left(\frac{\sqrt{x}(J_{1/3}(\zeta)+J_{-1/3}(\zeta))}{\sqrt{2}\lambda_n(J_{1/3}(\zeta_n)+J_{-1/3}(\zeta_n))}\right)dx\\ &= \frac{1}{\sqrt{2\lambda_n}\mathrm{Ai}(-\lambda_n)}\int_0^{\lambda_n}\mathrm{Ai}(-x)dx,\end{aligned}$$

where the second equality comes from (10.4.16) in [1]. Using now (4) and (10) in [2] with the same notations, we get

$$\begin{aligned}\int_{\lambda_n}^\infty \psi_n(y)dy \;&=\; \int_0^\infty\left(\frac{\sqrt{3x}K_{1/3}(\zeta)}{\pi\sqrt{2}\lambda_n(J_{1/3}(\zeta_n)+J_{-1/3}(\zeta_n))}\right)dx\\ &=\; \frac{1}{\sqrt{2\lambda_n}\mathrm{Ai}(-\lambda_n)}\int_0^\infty\left(\frac{\sqrt{x}K_{1/3}(\zeta)}{\pi\sqrt{3}}\right)dx \;=\; \frac{1}{3\sqrt{2\lambda_n}\mathrm{Ai}(-\lambda_n)}\end{aligned}$$

where $K_{1/3}$ is the Macdonald function of index 1/3, and the last equality stems from (10.4.82) in [1]. Putting everything together with (2) entails the formula

$$\mathbb{E}\left[\mathrm{e}^{-t||W||_1}\right] \;=\; \sum_{n\geq 1}\frac{1}{3\lambda_n\mathrm{Ai}(-\lambda_n)}\left(1+3\int_0^{\lambda_n}\mathrm{Ai}(-x)dx\right)\mathrm{e}^{-2^{-1/3}\lambda_n t^{2/3}} \qquad (3)$$

for every $t \geq 0$, which is up to some different notation exactly the one stated in [8].

Remarks (a) For conciseness, we did not detail the proofs of the results quoted from [2]. A perusal of pp. 583-584 therein shows that those results rely on completely classical considerations on Bessel functions. Recall that the method of [5] was a discretization of the problem with the help of a doubly exponential random walk, and several new theorems on Bessel functions taking eight pages of calculations, overall significantly more difficult to read. For this reason, although we used classical methods we thought that our simplification concerning this old result might still be useful.

(b) From the historical viewpoint the Feynman-Kac formula, which is of course a crucial point in the above computation, was actually obtained shortly after [5] in 1947. Notice that the simplified version which is used here hinges only upon the semi-group property of Brownian motion - see the proof of Theorem 5.8 in [6], and does not cover the full extent of the original Feynman-Kac formula. We refer to [4] and the references therein for a modern account on the latter and its applications to the computation of Laplace transforms of several Brownian additive functionals.

(c) The eigenelements $\{\psi_n^2, \lambda_n^2\}$ were computed by Titchmarsh in April 1940 in terms of the Hermite polynomials - see Section 4.2 in [8], and with the same argument as above one can show that this computation leads to the well-known Cameron-Martin formula

$$\mathbb{E}\left[\mathrm{e}^{-t||W||_2^2}\right] \;=\; \frac{1}{\sqrt{\cosh\sqrt{2t}}} \tag{4}$$

for all $t \geq 0$. However, the eigenelement argument turns out to be overall lenghtier than the two classical methods to obtain the Cameron-Martin formula, which rely respectively on the Karhunen-Loève decomposition and on Girsanov's transformation - see Example 3 in [4] for details, and much more on this topic.

(d) It is easy to see that the function $t \mapsto \mathbb{E}[\mathrm{e}^{-t||W||_1}]$ is analytic over $\mathbb{R}$. In [5], Kac raises the question how formula (3) should be continued to the negative real axis. To the best of our knowledge, this question remains open.

3 Series representations

With the help of the positive stable density of index 2/3, the Laplace transform (3) was inversed by Takács who gave a series representation for the density f_1 of $||W||_1$ (see Theorem 1 in [8]):

$$f_1(x) = \frac{2^{2/3}x^{-7/3}}{\sqrt{27\pi}} \sum_{n\geq 1} \frac{\lambda_n \mathrm{e}^{-2\lambda_n^3/27x^2}}{3\mathrm{Ai}(-\lambda_n)} \left(1 + 3\int_0^{\lambda_n} \mathrm{Ai}(-x)dx\right) U(1/6, 4/3, 2\lambda_n^3/27x^2)$$

where U is the usual confluent hypergeometric function. It is possible to inverse (2) for every $p > 0$ exactly the same way. For every $p > 0$, let g_p be the density of the positive stable law with index $2/(p+2)$ normalized such that for every $t \geq 0$,

$$\int_0^\infty g_p(x)\mathrm{e}^{-tx}dx \;=\; \mathrm{e}^{-t^{\frac{2}{p+2}}},$$

and set f_p for the density of $||W||_p$. By (2) and a change of variable we obtain

$$f_p(x) \;=\; \sum_{n\geq 1} \mu_n^p x^{p-1} g_p(x^p(\lambda_n^p)^{-(p/2+1)}) \qquad (5)$$

for every $x \geq 0$, with the notation

$$\mu_n^p \;=\; p(\lambda_n^p)^{-(p/2+1)}\psi_n^p(0)\left(\int_{\mathbb{R}} \psi_n^p(y)dy\right).$$

Let us consider the case $p = 2$. Changing the variable $z(x) = y(2^{1/4}x)$ in Equation (4.2.1) of [9], one gets

$$\lambda_n^2 \;=\; \frac{2n-1}{\sqrt{2}} \quad \text{and} \quad \psi_n^2(x) \;=\; \frac{\mathrm{e}^{-x^2/\sqrt{2}} H_n(2^{1/4}x)}{2^{\frac{n}{2}-\frac{1}{8}}(n!)^{\frac{1}{2}}\pi^{\frac{1}{4}}}$$

for every $n \geq 1$, where H_n stands for the $n-$th Hermite polynomial with physical notations, in other words the coefficient of t^n in the series expansion e^{2xt-t^2} :

$$\mathrm{e}^{2xt-t^2} \;=\; \sum_{n\geq 1} H_n(x)\frac{t^n}{n!}$$

for every $x, t \in \mathbb{R}$. Since on the other hand the density g_2 is explicit and equals

$$g_2(x) \;=\; \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-1/4x}x^{-3/2},$$

we obtain after some simple calculations the following expansion

$$f_2(x) \;=\; x^{-2}\sum_{p\geq 0} \frac{(-1)^p(2p)!(4p+1)}{\sqrt{\pi}2^{2p}(p!)^2}\mathrm{e}^{-(4p+1)^2/8x^2},$$

which could of course have been obtained in a shorter way, inverting directly the Cameron-Martin formula (4) and using the series expansion of $(1+t)^{-1/2}$. However, except in the cases $p = 1$ and $p = 2$, the coefficients μ_n^p cannot be made explicit in terms of usual special functions (see Lemma 4.2 in [4] and the discussion thereafter), although several variational formulæ for the eigenvalues λ_n^p can be derived (see [7] for an encyclopedic account).

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